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A synthesis procedure of distributed RC parameter networks

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A SYNTHESIS PROCEDURE OF DISTRIBUTED RC PARAMETER NETWORKS

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THESIS

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ABSTRACT

Synthesis procedures for driving-point functions and transfer functions in the w plane are developed in this thesis. For driving-point functions and transfer functions which are single-valued in s, where s is the complex frequency variable, the synthesis procedures were derived from the extension of the O'Shea's trasformation. For the transfer functions which satisfy the stability theorem in the w plane, a synthesis procedure is developed by using a distributed-lag element (DL) or a uniform distributed RC (UDRC) network with matched terminations.

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LIST OF SYMBOLS

A	Voltage transfer function
a	Scalar constant
C	Total capacitance
c	Per-unit capacitance
d	Total length
G	Transfer Function
j	$\sqrt{-1}$
P	Complex frequency variable in the P($=\cosh a\sqrt{s}$) plane
R	Total resistance
r	Per-unit resistance
s	Complex frequency variable in the s plane
w	Complex frequency variable in the w($=e^{a\sqrt{s}}$) plane
ω	Angular frequency variable in the s plane
Y	Driving-point admittance function
Z	Driving-point impedance function

I. INTRDUCTION

A. Purpose of Thesis

Synthesis of uniformly distributed RC driving-point functions based on Richards' transformation (1) and O'Shea's transformation (2) are restricted to certain kinds of networks with single-valued functions of s, in which s is the complex frequency variable. Extentions to the transfer-function synthesis of uniformly distributed RC networks are also restricted to certain kinds of networks with single-valued functions of s (3).

This paper presents procedures of synthesis for more general functions based on the frequency variable $w = e^{a\sqrt{s}}$ in which a is a constant.

A series of frequency responses are computer-plotted for various pole locations within the w-plane region of bounded input-bounded output stability (4). These curves can be used to obtain the optimum pole locations for synthesis and design requirements.

B. Review of the Literature

Other articles dealing with distributed network synthesis have been published:

Richard presented a frequency transformation,

$$S(s) = \tanh\ (\ as\ /\ 2\ ) = \frac{e^{as} - 1}{e^{as} + 1}$$

and Tachibana used a variation of Richard's transformation:

$$\lambda = \tanh a\sqrt{s},$$

where $a = \sqrt{rcd^2} = \sqrt{RC}$ = a constant,

to synthesize distributed RC networks.

O'Shea presented another frequency transformation,

$$P = \cosh a\sqrt{s}.$$

After the two-port open-circuit impedance parameters of a uniformly distributed RC network have been multiplied by $a\sqrt{s}\,\sinh a\sqrt{s}$, we get rational impedance parameters in the P plane:

$$\begin{bmatrix} Z_{ij}^{P} \end{bmatrix} = R \begin{bmatrix} P & 1 \\ 1 & P \end{bmatrix}.$$

Bourquin presented a third frequency transformation,

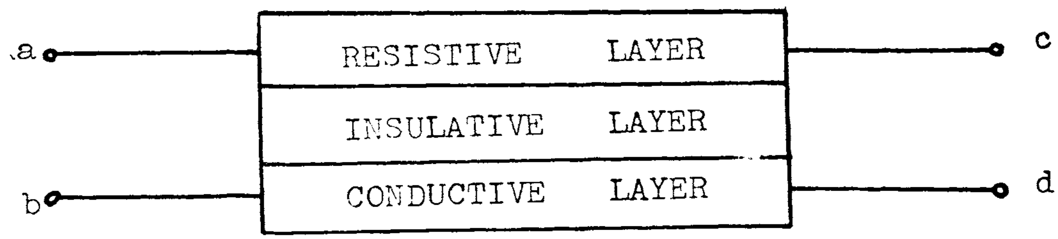
$$w = e^{a\sqrt{s}},$$

in discussing the stability of a class of lumped-distributed systems (4).

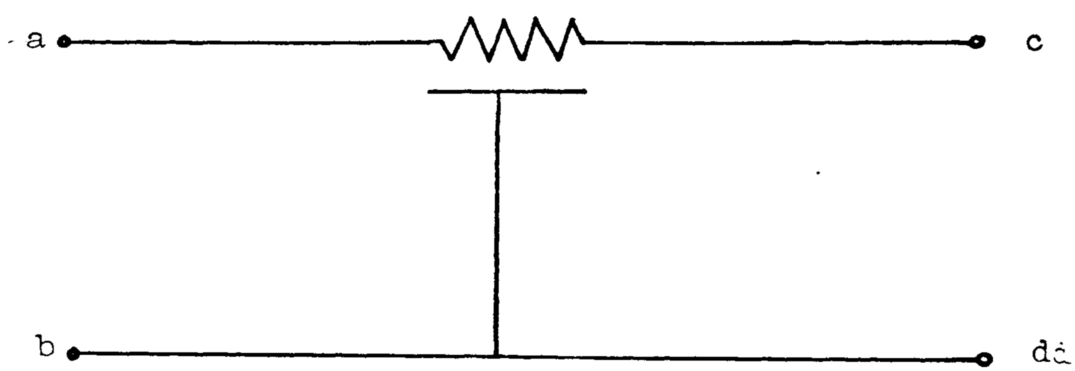
C. Model of a Uniform Distributed RC Network

Physically, the general form of a uniformly distributed network is shown in Figure 1-1 (a). Symbolically, the uniformly distributed network is represented by Figure 1-1 (b), where the straight line under R indicates a uniformly distributed RC network.

The two-port open-circuit impedance parameters and the short-circuit admittance parameters of this network are:



(a) Line structure



(b) Symbol

Figure 1-1: Structure and Symbol for the uniform RC line.

$$[Z] = \begin{bmatrix} \frac{R}{a\sqrt{s}} \coth a\sqrt{s} & \frac{R}{a\sqrt{s}} \operatorname{csch} a\sqrt{s} \\ \frac{R}{a\sqrt{s}} \operatorname{csch} a\sqrt{s} & \frac{R}{a\sqrt{s}} \coth a\sqrt{s} \end{bmatrix} \tag{1-1}$$

$$[Z] = \begin{bmatrix} \frac{a\sqrt{s}}{R} \coth a\sqrt{s} & -\frac{a\sqrt{s}}{R} \operatorname{csch} a\sqrt{s} \\ -\frac{a\sqrt{s}}{R} \operatorname{csch} a\sqrt{s} & \frac{a\sqrt{s}}{R} \coth a\sqrt{s} \end{bmatrix} \tag{1-2}$$

II. SINGLE-VALUED FUNCTION SYNTHESIS

A. Single-valued Driving-point Function Synthesis of Networks Having a Prescribed Function

A theorem developed by O'Shea states (2):

A necessary and sufficient condition that a function Z(s) be the driving-point impedance function of a network composed of uniform RC lines of the type shown in Figure 1-1 is that Z(s) consist of a real rational function of P,

$$Z(P) = K \frac{\prod_{j=1}^{N+1}(P + b_j)}{\prod_{i=1}^{N}(P + a_i)} . \qquad (2\text{-}1)$$

K is a positive number, multiplied by

$$\frac{1}{\sqrt{sRC}\ \sinh\sqrt{sRC}} .$$

The poles and zeros of Z(P) have the following properties:

1) The poles and zeros are simple and are interlaced along the real axis of the P plane.
2) The most negative and most positive critical points are zeros.
3) All of the poles and zeros are less than, or equal to, one in magnitude.

As an extension of this theorem, a theorem in the w plane can be established.

Theorem: A necessary and sufficient condition that a function Z(s) be the driving-point impedance function of a network composed of uniform RC lines of the type shown in Figure 1-1 is that Z(s) consist of a real rational function of w,

$$Z(w) = K \frac{\prod_{i=1}^{N+1} (w^2 + 2\ b_i w + 1)}{w \prod_{j=1}^{N} (w^2 + 2\ b_j w + 1)} \quad . \tag{2-2}$$

K is a positive number, multiplied by $w \,/\, (\sqrt{s\ R\ C}\ (w^2 - 1))$.

The poles and zeros of w Z(w) are simple and are interlaced along the unit circle centered at origin in the w plane, except for the critical points nearest or on the $(w=) \pm 1$ points which are zeros.

Proof:

First, the necessity will be demonstrated. O'Shea's realizability conditions are illustrated in Figure 2-1 (a). For each real P,

$$P = \cosh a\sqrt{s} = \frac{w + \frac{1}{w}}{2}, \qquad |P| \leq 1, \tag{2-3}$$

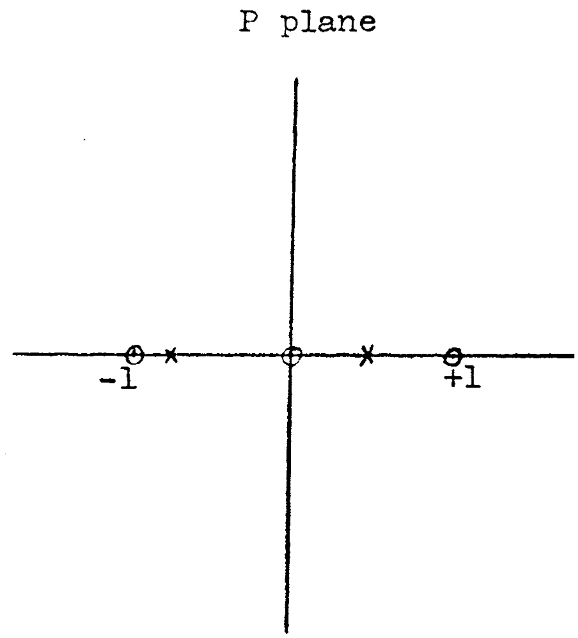
$$w = P \pm j\sqrt{1 - P^2}; \tag{2-4}$$

hence, $|w|^2 = P^2 + (1 - P^2) = 1,$ (2-5)

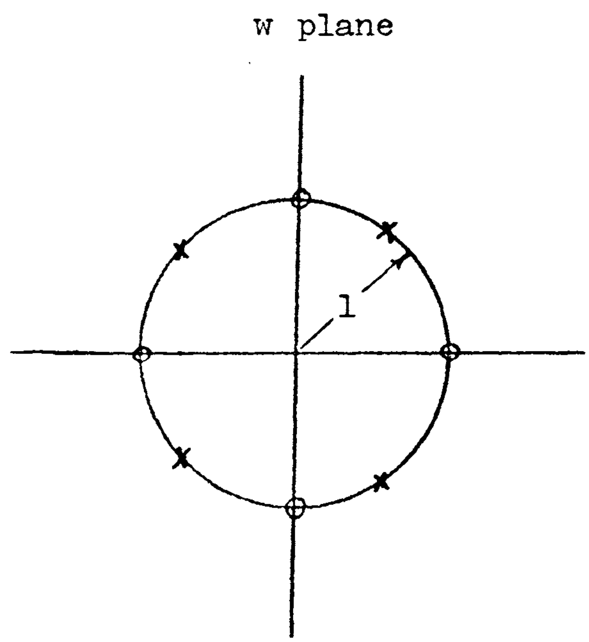
indicating the conjugate points on the unit circle centered at origin in the w plane, as shown in Figure 2-1 (b).

In the P plane, the most negative and most positive points are zeros. By equation (2-4) the real part of w is equal to P. This proves that the critical points nearest or on the $(w=) \pm 1$ points are zeros.

Next the sufficiency will be demonstrated: From the above theorem, it is seen that a physically realizable network comprised of $\overline{\text{URC}}$ networks with equal RC products can be expressed as:



(a) Points defined in the P plane



(b) Mapped into the w plane

Figure 2-1: Five points defined in the P plane and how they map into the w plane.

$$Z(w) = K\,\frac{(w^2-2Z_1w+1)\,.....\,(w^2-2Z_{N+1}w+1)}{w(w^2-2P_1w+1)\,....\,(w^2-2P_Nw+1)} \tag{2-6}$$

where $K \geq 0$ and

$$-1 \leq Z_1 < P_1 < Z_2 < P_2 \,..... \quad P_N < Z_{N+1} \leq 1 \tag{2-7}$$

The procedure is to use a partial-fraction expansion of $\left(\frac{2w}{w^2-1}\right)^2 Z(w)$ as follows:

$$\left(\frac{2w}{w\ -1}\right)^2 Z(w)^2 = \frac{2k_1w}{(w-1)^2} + \frac{2k_2w}{(w+1)^2} + \sum_{j=1}^{N} \frac{2h_jw}{w^2-2P_jw+1} \tag{2-8}$$

or

$$Z(w) = \frac{k_1(w+1)^2}{2w} + \frac{k_2(w+1)^2}{2w} + \sum_{j=1}^{N} \frac{h_j(w^2-1)^2}{2w(w^2-2P_jw+1)}. \tag{2-9}$$

The constants k_1, k_2, h_1, h_2, $.....h_N$ are always nonnegative, and the realization for each term of $Z(w)$ is always possible.

Consider k_1:

$$\begin{aligned} k_1 &= \left(\frac{(w-1)^2}{2w}\right)\left(\frac{2w}{w^2-1}\right)^2 Z(w)\Big|_{w=1} \\ &= K\,\frac{2(w^2-2Z_1w+1)\,.....\,(w^2-2Z_{N+1}w+1)}{(w+1)^2(w^2-2P_1w+1)\,.....\,(w^2-2P_Nw+1)}\Bigg|_{w=1} \\ &= K\,\frac{(1-Z_1)\,.....\,(1-Z_{N+1})}{(1-P_1)\,.....\,(1-P_N)}. \end{aligned} \tag{2-10}$$

However, $Z_i \leq 1, \quad i = 1,2,\ \ldots\ldots\ldots,\ N+1$

$$P_i \leq 1, \quad i = 1,2,\ \ldots\ldots\ldots,\ N; \tag{2-11}$$

hence, $k_1 \geq 0$.

Now consider h_j:

$$\begin{aligned} h_j &= \left(\frac{w^2-2P_jw+1}{2w}\right)\left(\frac{2w}{w^2-1}\right)^2 Z(w)\Bigg|_{w=P_j\pm j\sqrt{1-P_j^2}} \\ &= 2K\,\frac{(w^2-2Z_1w+1)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}{(w^2-1)^2(w^2-2P_1w+1)\ ..\ (w^2-2P_{j-1}w+1)(w^2-2P_{j+1}w+1)} \end{aligned}$$

$$\left.\cdot\frac{\cdots\ (w^2-2Z_{N+1}w+1)}{\cdots\ (w^2-2P_Nw+1)}\right|_{w=P_j\pm j\sqrt{1-P_j^2}}$$

$$= 2K\,\frac{\left(\frac{w^2+1}{2w}-Z_1\right)\ \cdots\cdots\cdots\cdots\cdots\cdots}{\left(\left(\frac{w^2+1}{2w}\right)^2-1\right)\left(\frac{w^2+1}{2w}-P_1\right)\cdots\left(\frac{w^2+1}{2w}-P_{j-1}\right)\left(\frac{w^2+1}{2w}-P_{j+1}\right)}$$

$$\left.\cdot\frac{\cdots\ \left(\frac{w^2+1}{2w}-Z_{N+1}\right)}{\cdots\ \left(\frac{w^2+1}{2w}-P_N\right)}\right|_{w=P_j\pm j\sqrt{1-P_j^2}}$$

$$= 2K\,\frac{(P_j-Z_1)\ \cdots\cdots\cdots\cdots\ (P_j-Z_{N+1})}{(P_j-1)(P_j-P_1)\cdots(P_j-P_{j-1})(P_j-P_{j+1})\cdots(P_j-P_N)}. \tag{2-12}$$

Since the number of factors with negative signs in the numerator and denominator are equal, the h_j's are non-negative. The same argument applies to K_2.

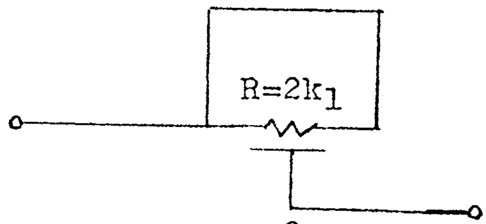
From O'Shea's work (2) the realization in the P domain for each term of

$$Z(P) = k_1(P+1) + k_2(P-1) + \sum_{j=1}^{N}\frac{h_j(P+1)(P-1)}{(P-P_j)} \tag{2-13}$$

is shown in Figure 2-2 (a), (b), (c). Since the coefficients of corresponding terms in equations (2-9) and (2-13) are equal, the realization in the w domain for equation (2-9) is also the same as shown in Figure 2-2 (a), (b), (c). Thus the sufficiency condition has been established.

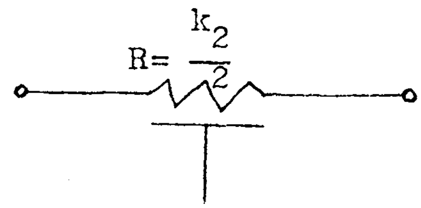
Example 1:

Synthesize a network having the following driving-point function:



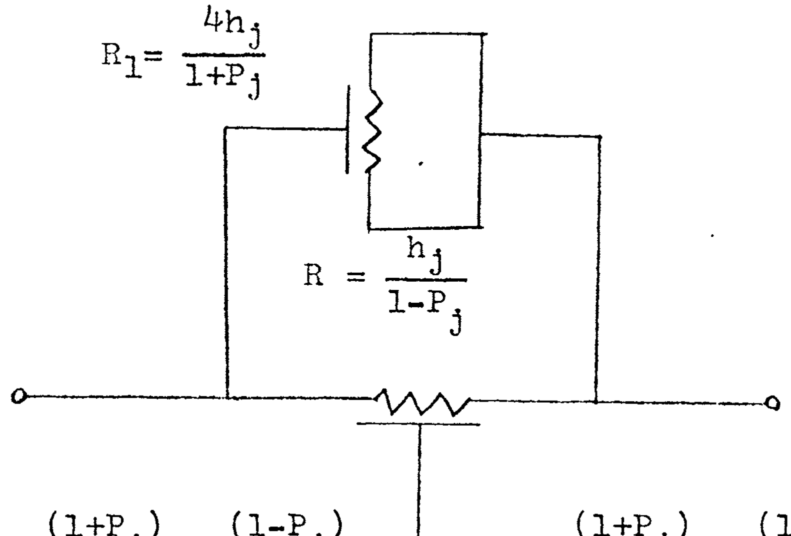
(a) $Z(w) = k_1\frac{(w+1)^2}{2w}$

or $Z(P) = k_1(P+1)$



(b) $Z(w) = k_2\frac{(w-1)^2}{2w}$

or $Z(P) = k_2(P-1)$



(c) $Y(w) = \dfrac{\frac{(1+P_j)}{2h_j}}{\frac{(w+1)^2}{2w}} + \dfrac{\frac{(1-P_j)}{2h_j}}{\frac{(w-1)^2}{2w}}$ or $Y(P) = \dfrac{\frac{(1+P_j)}{2h_j}}{P+1} + \dfrac{\frac{(1-P_j)}{2h_j}}{P-1}$

Figure 2-2: The realization in the w domain and in the P domain for each term of equations (2-9) and (2-13).

$$Z(s) = \frac{(\tanh\sqrt{s})(\tanh^2\sqrt{s} + 4)}{\sqrt{s}(\tanh^2\sqrt{s}+1)(\tanh^2\sqrt{s}+9)}. \qquad (2\text{-}14)$$

Making the transformation in the w plane,

$$Z(w) = Z(s)\frac{\sqrt{s}(w^2-1)}{2w}\Bigg|_{w=e^{\sqrt{s}}} = \frac{(w^2+1)(w^4-2w^2+1)(5w^4+6w^2+5)}{2w(2w^2+2)(10w^4+16w^2+10)}.$$

Now form $Z(w)(\frac{2w}{w^2-1})^2$ and expand it in a partial-fraction expansion:

$$Z(w)\left(\frac{2w}{w^2-1}\right)^2 = \frac{(w^2+1)(w^4+\frac{6}{5}w^2+1)w}{2(w^4+1)(w^4+\frac{8}{5}w^2+1)}$$

$$= \frac{\frac{3w}{16}}{w^2-\sqrt{2}w+1} + \frac{\frac{3w}{16}}{w^2+\sqrt{2}w+1} + \frac{\frac{w}{16}}{w^2-\sqrt{\frac{2}{5}}w+1} + \frac{\frac{w}{16}}{w^2+\sqrt{\frac{2}{5}}w+1}.$$

Multiply both sides by $(\frac{w^2-1}{2w})^2$ to obtain:

$$Z(w) = \left(\frac{w^2-1}{2w}\right)^2\left(\frac{\frac{3w}{16}}{w^2-\sqrt{2}w+1} + \frac{\frac{3w}{16}}{w^2+\sqrt{2}w+1} + \frac{\frac{w}{16}}{w^2-\sqrt{\frac{2}{5}}w+1} + \frac{\frac{w}{16}}{w^2+\sqrt{\frac{2}{5}}w+1}\right).$$

Each term is realized as in Figure 2-2, and thus the RC network realization for equation (2-14) is shown in Figure 2-3.

B. Single-valued Transfer-function Synthesis Using an NIC and One-port RC Networks.

Consider the circuit proposed by Yanagisawa (5) as shown in Figure 2-4. Analysis of the network yields the following voltage transfer function:

$$A = \frac{V_o}{V_{in}}\Bigg|_{I_2=0} = \frac{Y_{1b}-Y_{1a}}{(Y_{1b}-Y_{1a})+(Y_{2b}-Y_{2a})}. \qquad (2\text{-}15)$$

Analysis of the URC network transfer function in the P plane yields the properties of the single-valued voltage transfer function in the w plane:

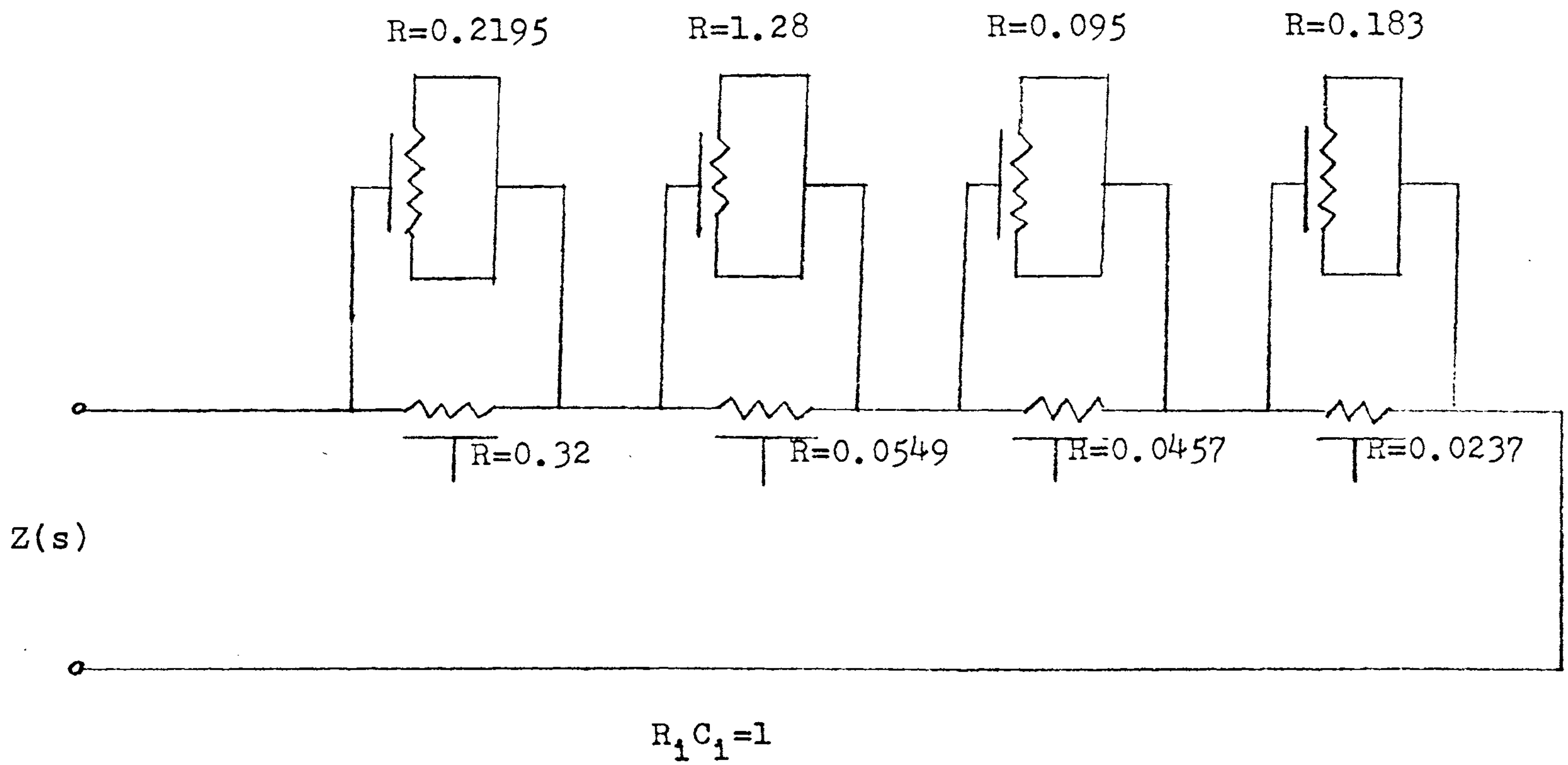


Figure 2-3: The realization in the w domain with the driving-point function in Example 1.

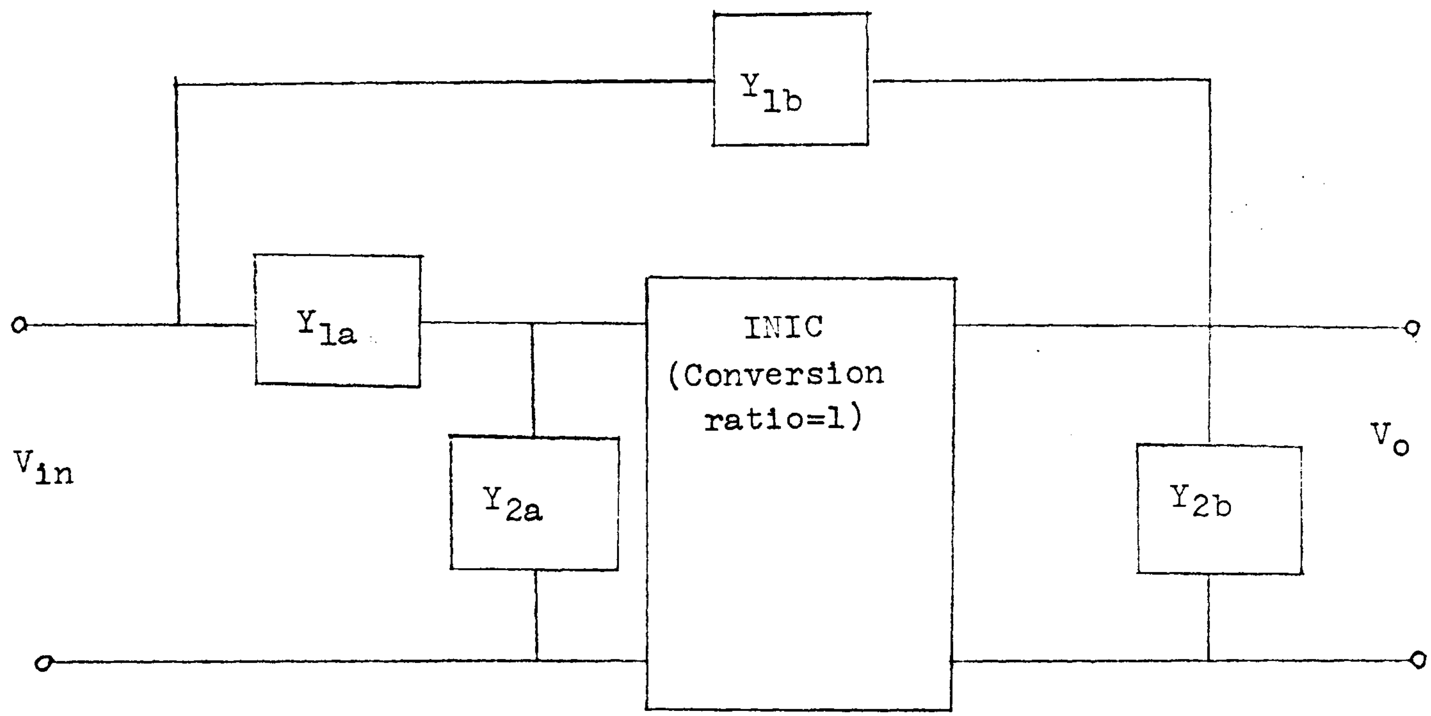


Figure 2-4: NIC-RC configuration.

$$A(w) = K\frac{w^{N_1}(w^{2N_2}+a_1w^{2N_2-1}+ \,..\, +a_{2N_2-1}w+a_{2N_2})}{w^{2M}+b_1w^{2M-1}+ \,..\, +b_{2M-1}w+b_{2M}} \tag{2-16}$$

where

$$N_1 + N_2 = M \tag{2-17}$$

$$a_{2N_2} = 1 = b_{2M}$$

$$a_1 = a_{2N_2-1} \qquad b_1 = b_{2M-1}$$

$$a_2 = a_{2N_2-2} \qquad b_2 = b_{2M-2}$$

$$\vdots \qquad\qquad \vdots$$

$$a_K = a_{2N_2-K} \qquad b_K = b_{2M-K}\ , \tag{2-18}$$

Proof:

Let the voltage transfer function A(P) be (2)

$$A(P) = \frac{C(P)}{D(P)} = \frac{C_0+C_1P+C_2P^2+ \,..\, +C_nP^n}{D_0+D_1P+D_2P^2+ \,..\, +D_dP^d}\ . \tag{2-19}$$

A conclusion of O'Shea states that for stability consideration, the degree of the numerator be no greater than the degree of the denominator.

Since

$$P = \frac{w^2+1}{2w}$$

$$A(w) = A(P)\Big|_{P=\frac{w^2+1}{2w}} = \frac{C'(w)}{D'(w)}$$

$$A(w) = \frac{(2w)^{(d-n)}(C_0(2w)^n+C_1(2w)^{n-1}(w^2+1)+C_2(2w)^{n-2}(w^2+1)^2}{D_0(2w)^d+D_1(2w)^{d-1}(w^2+1)+D_2(2w)^{d-2}(w^2+1)^2}$$

$$\cdot\ \frac{+\ldots +C_n(w^2+1)^n)}{+ \,..\, +D_d(w^2+1)^d}\ . \tag{2-20}$$

Compare equation (2-20) with equation (2-16):

$d - n = N_1, \quad n = N_2, \quad d = M$

and

$N_1 + N_2 = d - n + n = M$

This proves the property $N_1 + N_2 = M$.

The coefficient conditions are proved by the induction method. From equation (2-20),

$$(2w)^n C'(w) = C_0(2w)^n + C_1(2w)^{n-1}(w^2+1) + .. + C_n(w^2+1)^n$$

For n=1,

$$(2w)C'(w) = C_0(2w) + C_1(w^2+1) = C_1(w^2+\frac{2C_0}{C_1}w+1)$$

For n=2,

$$(2w)C'(w) = C_0(2w)^2 + C_1(2w)(w^2+1) + C_2(w^2+1)^2$$

$$= C_2(w^4 + \frac{2C_1}{C_2}w^3 + \frac{2C_0+2C_3}{C_2}w^2 + \frac{2C_1}{C_2}w + 1).$$

Both of these cases satisfy the coefficient conditions. Now assuming that these conditions are also satisfied for n=K,

$$(2w)^K C'(w) = C_0(2w)^K + C_1(2w)^{K-1}(w^2+1) + .. + C_K(w^2+1)^K.$$

$$= C_K(w^{2K} + a_1 w^{2K-1} + .. + a_{2n-1}w + a_{2K})$$

$$= C_K(w^{2K} + a_1 w^{2K-1} + .. + a_1 w + 1) .$$

For n=K+1,

$$(2w)^{K+1}C'(w) = C_n(w^{2K} + a_1 w^{2K-1} + .. + a_1 w + 1)(2w)$$

$$+ C_{K+1}(w^2+1)^{K+1} .$$

By the symmetrical property of the binomial expansion of the (K+1)th degree term, this expansion adds the same magnitude to the coefficients of corresponding terms of the former part, and the coefficient conditions still hold true. This completes the proof. A similar proof is applicable to D'(w).

For stability reasons the poles of A(w) must be restricted. A theorem Bourquin developed states that for the bounded input-bounded output stability of a system for which the transfer function is a rational function of only $w=e^{a\sqrt{s}}$, when the degree of the numerator is not greater than the degree of the denominator, it is necessary and sufficient that all poles of the transfer function occur within that region of the w plane bounded by $e^{|\phi|+j\phi}$, where $-\pi\leq\phi=\arg(w)\leq\pi$, as shown in Figure 2-5.

The synthesis procedure is substantially the same as in the lumped RC case except that it is in the w plane rather than the s plane.

The degree of freedom in the choice of roots can be used to obtain a realization that provides the best compromise between minimum difference of resistance values and minimum sensitivity of A due to the changes of poles in the w palne.

The synthesis procedure is described as follows: Consider the transfer function specified in equation (2-16):

$$A(w) = K \frac{w^{N_1}(w^{2N_2}+a_1w^{2N_2-1}+ \,..\, +a_{2N_2})}{w^{2M}+b_1w^{2M-1}+ \,..\, +b_{2M}} \, .$$

Choose a polynomial Q(w) with distinct complex conjugate roots of unity magnitude and of degree at least an even degree larger than the denominator. This polynomial

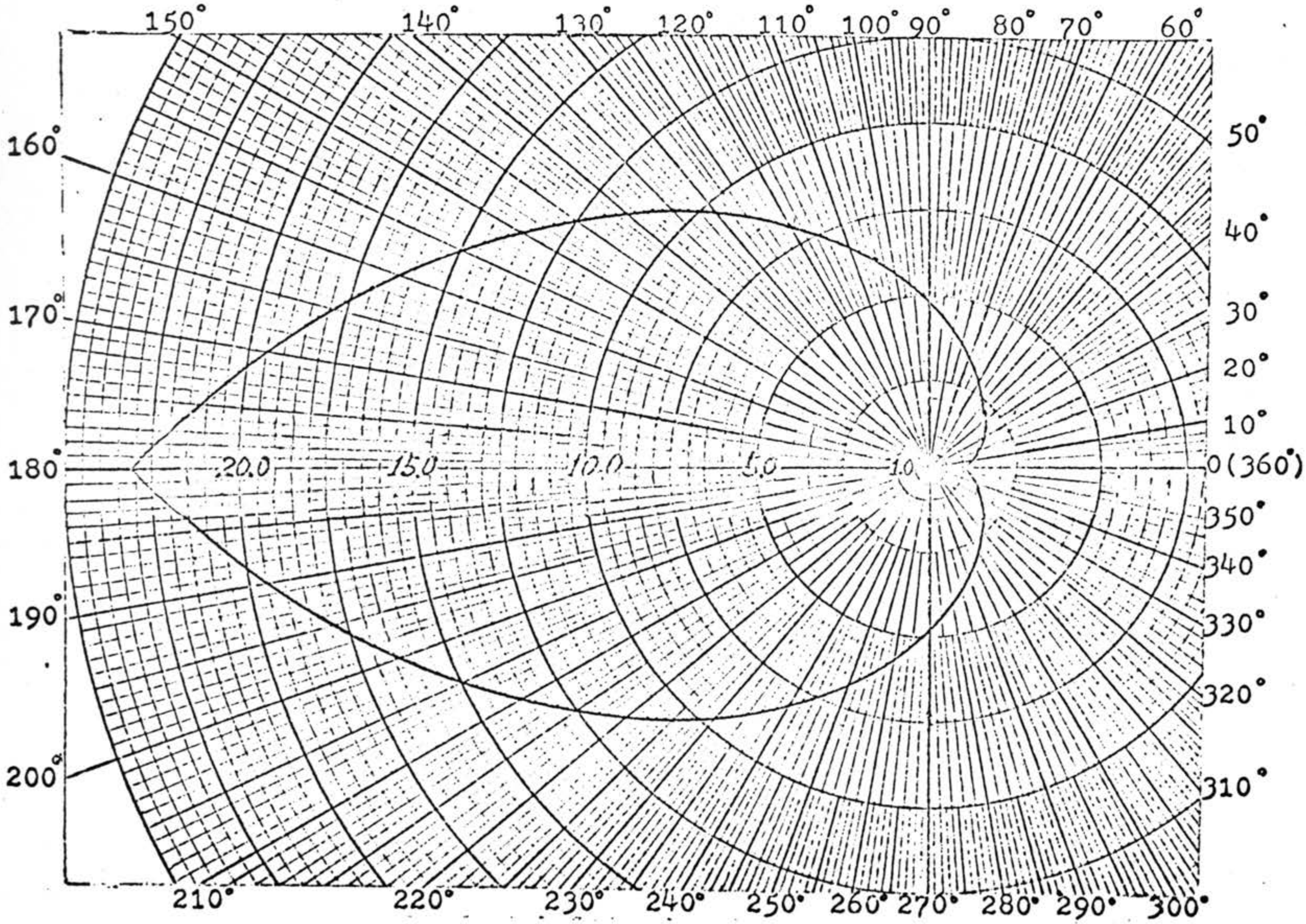


Figure 2-5: The region of stability for functions rational in w (stable inside the curve).

divided by w^T is a single-valued function of the complex frequency s. Let

$$Q(w) = \prod_{i}^{T}(w^2 - 2P_i w + 1), \quad T \geq M + 1 \tag{2-21}$$

$$P(w) = \frac{Q(w)}{(w)^{T-M}} . \tag{2-22}$$

Divide the numerator and the denominator of equation (2-16) by equation (2-22), as follows:

$$A(w) = \frac{N(w)}{D(w)} = \frac{\frac{N(w)}{P(w)}}{\frac{N(w)}{P(w)} + \frac{D(w)-N(w)}{P(w)}} . \tag{2-23}$$

Comparison of equations (2-23) and (2-15) suggests the following identifications:

$$Y_{1b} - Y_{1a} = \frac{N(w)}{P(w)} = \sum_{i=1}^{T} \frac{k_i w}{w^2-2P_i w+1} \tag{2-24}$$

$$Y_{2b} - Y_{2a} = \frac{D(w)-N(w)}{P(w)} = \sum_{i=1}^{T} \frac{h_i w}{w^2-2P_i w+1} . \tag{2-25}$$

In equations (2-24) and (2-25) the various residues will be real and either positive or negative depending on the coefficients of N(w) and D(w) and the choice of zeros of P(w). Thus the individual one-port realizations are obtained directly in Foster form.

Example 2:

Realize the open-circuit voltage ratio,

$$A(w) = \frac{(w-1)^2(w^2+4w+1)}{w^4+2w^3+6w^2+2w+1} . \tag{2-26}$$

Choose

$$Q(w) = (w^2+1)(w-1)^2(w+1)^2;$$

then

$$P(w) = \frac{(w^2+1)(w-1)^2(w+1)^2}{2w}$$

gives

$$Y_{1b} - Y_{1a} = \frac{N(w)}{P(w)} = \frac{2w(w-1)^2(w^2+4w+2)}{(w^2+1)(w-1)^2(w+1)^2}$$

$$= \frac{4w}{w^2+1} - \frac{2w}{(w+1)^2}.$$

Hence,

$$Y_{1b} = \frac{4w}{w^2+1}, \quad Y_{1a} = \frac{2w}{(w+1)^2}.$$

Also,

$$Y_{2b} - Y_{2a} = \frac{D(w)-N(w)}{P(w)} = \frac{24w^3}{(w^2+1)(w-1)^2(w+1)^2}.$$

$$= \frac{-6w}{w^2+1} + \frac{3w}{(w+1)^2} + \frac{3w}{(w-1)^2}.$$

Hence,

$$Y_{2b} = \frac{3w}{(w+1)^2} + \frac{3w}{(w-1)^2}, \quad Y_{2a} = \frac{6w}{w^2+1}.$$

The distributed RC network realization for Equation (2-26) is shown in Figure 2-6.

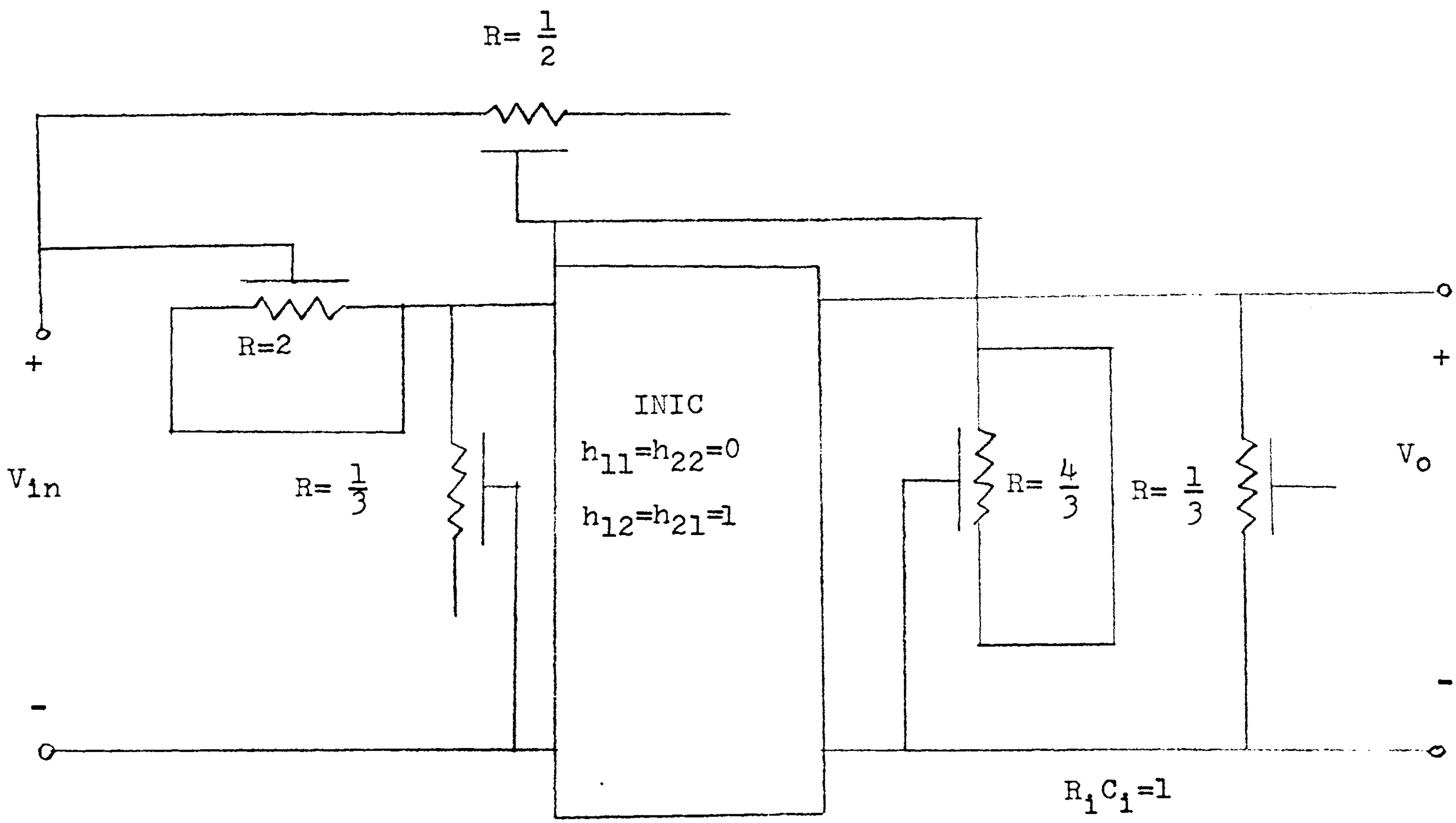


Figure 2-6: NIC-RC configuration of Example 2.

III. DOUBLE-VALUED FUNCTION SYNTHESIS OF NETWORKS HAVING A PRESCRIBED TRANSFER FUNCTION

A. Double-valued Transfer Functions of s For Which the Transfer Function is a Rational Function of $w=e^{a\sqrt{s}}$ Only.

Since $w=e^{a\sqrt{s}}$, this function is double-valued in s. For each value of s,

$$s=re^{j(\theta+2n\pi)}, \qquad n=0,\ 1,\ 2,\ldots,$$

as shown in Figure 3-1(a) in the s plane.

Since

$$s=\sqrt{r}e^{j\frac{(\theta+2n\pi)}{2}}, \qquad n=0,\ 1,\ 2,\ldots,$$

there are two corresponding points in the $\sqrt{s}$ plane for each point in the s plane, as shown in Figure 3-1(b).

Since $w=e^{a\sqrt{r}e^{j\frac{(\theta+2n\pi)}{2}}}$, $\qquad n=0,\ 1,\ 2,\ldots,$

there are two points in the w plane for each point in the s plane, as shown in Figure 3-1(c).

The rational functions A(w) which cannot satisfy the requirements for the single-valued function defined in section II-B are classified as double-valued functions.

B. Double-valued Transfer-function Synthesis Based on the w Transformation.

In the double-valued transfer-function synthesis a distributed-lag element (DL), or a uniform distributed RC network (UDRC) with matched terminations, is cascaded with a frequency invariant gain K. This combination, incorporated in a closed loop with positive and negative feedback as shown in Figure 3-2, can be used as a basic

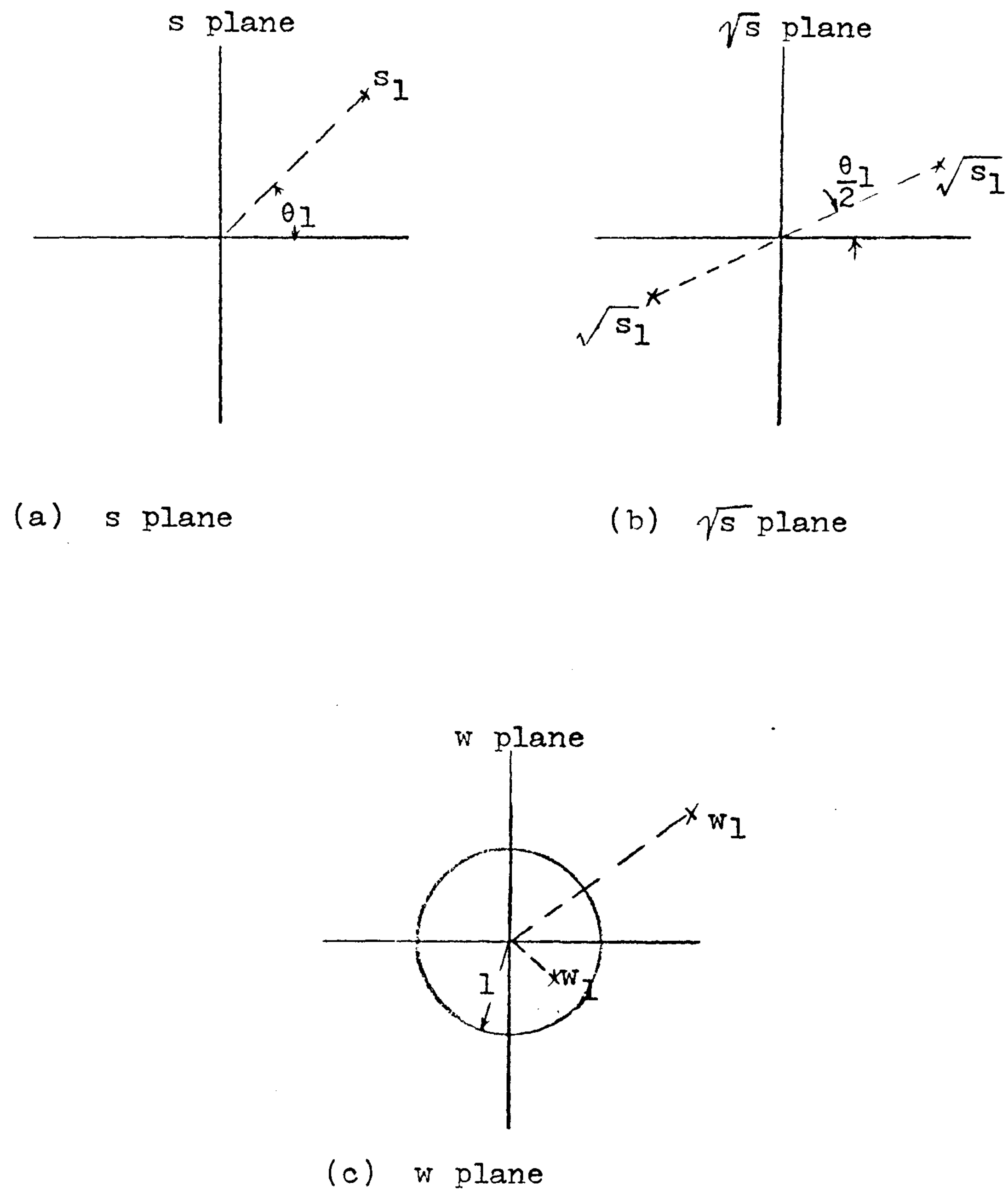


Figure 3-1: The transformation $w=e^{a\sqrt{s}}$.

network. The overall transfer function of the system is

$$A(w) = \frac{K}{w \pm K} \tag{3-1}$$

which has a simple pole at $\pm K$.

The feedback systems used in this thesis may be diagrammed as shown in Figure 3-3. However, the transfer functions G_{23} or G_{34}, either or both, may contain inner feedback loops. The overall transfer function of the system is

$$A(w) = \frac{G_{23}}{1 \pm G_{23}G_{34}}. \tag{3-2}$$

Example 3:

Realize the voltage transfer function

$$A = \frac{w(w^2+2)}{w^3+6w^2+11w+6}. \tag{3-3}$$

Expand it in a factored form:

$$A = \frac{w(w^2+2)}{(w+1)(w+2)(w+3)} = A_1A_2,$$

where

$$A_1 = \frac{w}{w+1}$$

and

$$A_2 = \frac{w^2+2}{(w+2)(w+3)}.$$

Expand A_2 in a partial fraction expansion:

$$A_2 = 1 + \frac{6}{w+2} - \frac{11}{w+3}.$$

Thus, the RC network realization for equation (3-3) is shown in Figure 3-4.

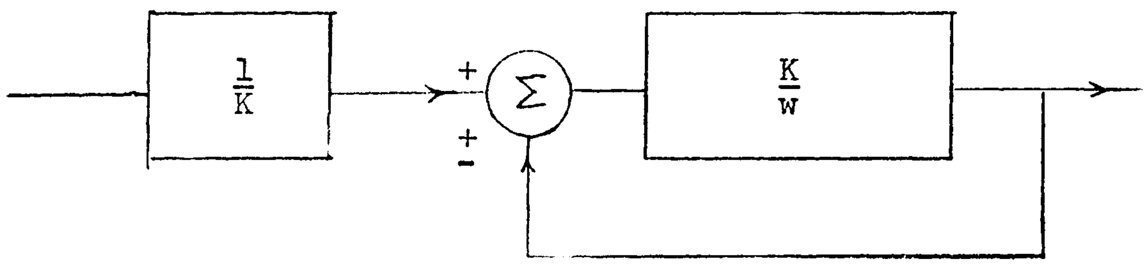


Figure 3-2: Block diagram for $F(w)= \frac{1}{w \pm K}$.

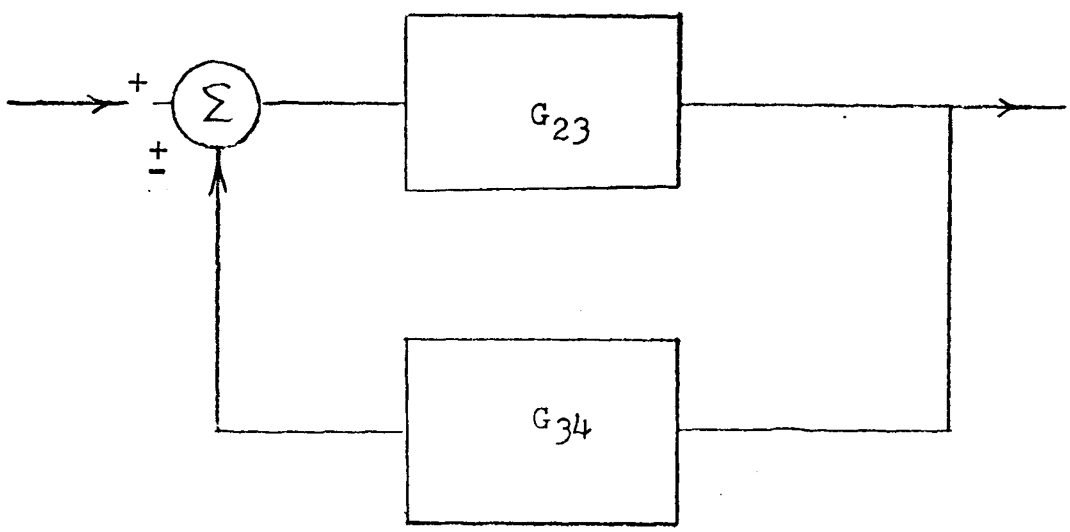


Figure 3-3: Block diagram for $A(w)= \frac{G_{23}}{1 \pm G_{23}G_{34}}$.

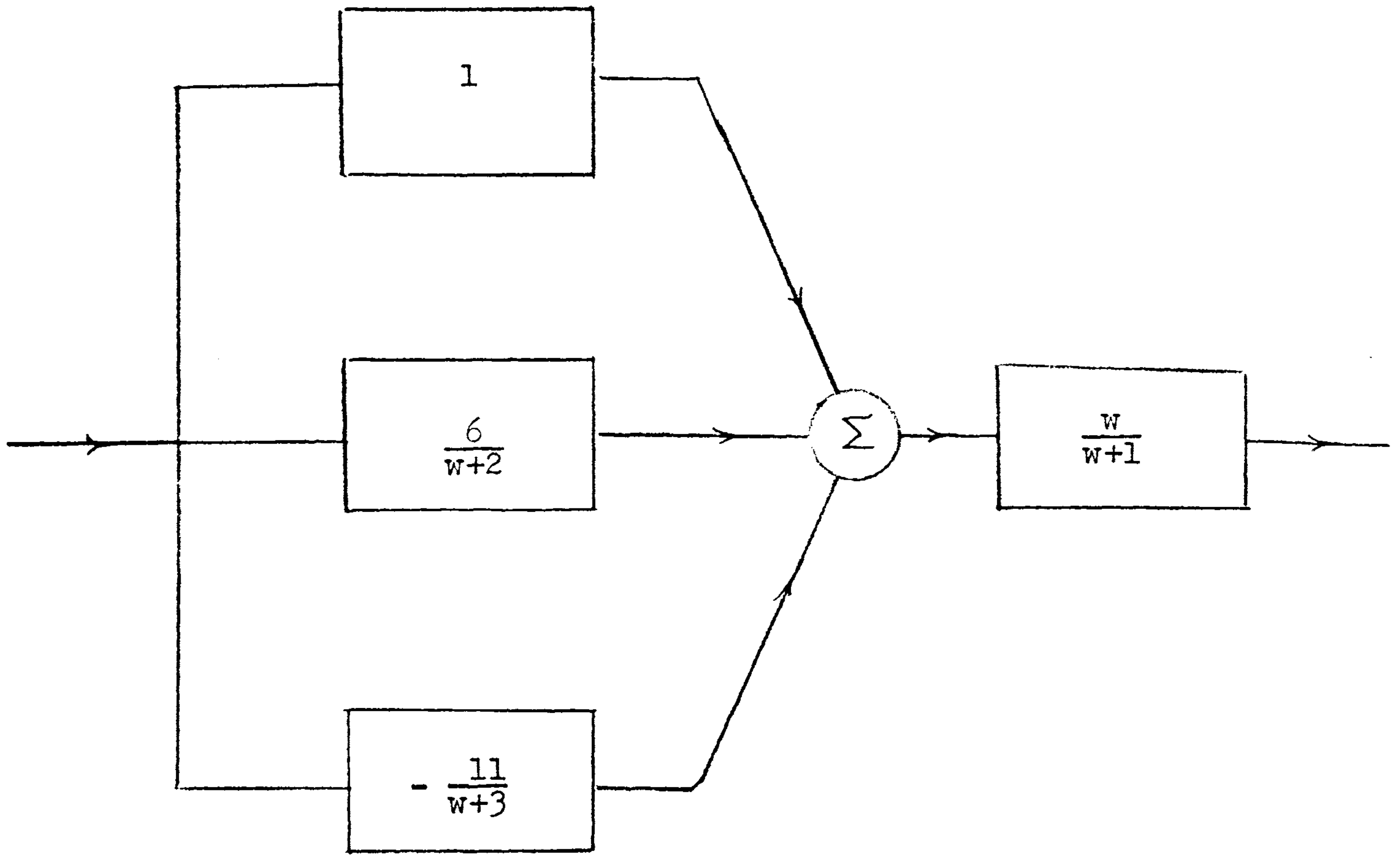


Figure 3-4: Block diagram for the transfer function of Example 3.

IV. CONCLUSIONS

The w plane provides a convenient plane in which to synthesize a system for which the transfer function in w is rational.

The function A(P), which is a rational function of P, should be a single-valued function of s. The function A(w), which is a rational function of w, can be either a single-valued function or a double-valued function of s.

Based on the w transformation, the transfer functions which satisfy the stability theorem can be synthesized as shown in section III-B. The decomposition of the numerator and denominator of the transfer function as presented in this paper is not unique, due to many possible choices of Q(w). A series of plots of frequency response in the Appendix can be used as a reference to choose the optimum poles of A(w).

The synthesis procedure for any double-valued driving-point function has not been covered.

APPENDIX: THE PLOTS OF FREQUENCY RESPONSE FOR TRANSFER FUNCTIONS RATIONAL IN W

For a rational transfer function which satisfies the stability theorem it is sufficiently general to investigate the frequency response of $F(w) = \frac{1}{w-a}$ in order to determine the frequency response of $A(w)$.

The magnitudes of the frequency response of $F_1(w) = \frac{(1-a)(1-a^*)}{(w-a)(w-a^*)}$ and $F_2(w) = \frac{w(1-a)(1-a^*)}{(w-a)(w-a^*)}$ are plotted by choosing several locations within the w-plane region of bounded input-bounded output stability, as listed in tables A-1 and A-2. In Figures A-1 through A-13 the ordinate represents $T_1(= 20\log_{10} |F_1(w)|)$. In Figures A-14 throught A-26 the ordinate represents $T_2(= 20\log_{10} |F_2(w)|)$. The abscissa represents the real frequency in the logarithmic scale.

figure number	pole angle	pole magnitude being plotted $\lvert a \rvert$
A- 1	180°	0.0;2.0;4.0;6.0;8.0;10.0;12.0;14.0;16.0;18.0;20.0;22.0;
A- 2	165°	0.0;2.0;4.0;6.0;8.0;10.0;12.0;14.0;16.0;
A- 3	150°	0.0;2.0;4.0;6.0;8.0;10.0;12.0;
A- 4	135°	0.0;2.0;4.0;6.0;8.0;10.0;
A- 5	120°	0.0;2.0;4.0;6.0;8.0;
A- 6	105°	0.0;2.0;4.0;6.0;
A- 7	90°	0.0;1.0;2.0;3.0;4.0;
A- 8	75°	0.0;1.0;2.0;3.0;
A- 9	60°	0.0;1.0;2.0;
A-10	45°	0.0;1.0;2.0;
A-11	30°	0.00; 0.25;0.50;0.75;1.00;1.25;1.50;
A-12	15°	0.00;0.25;0.50;0.75;1.00;
A-13	0°	0.00;0.25;0.50;0.75;

Table A-1: List of pole magnitudes plotted for the frequency response of T_1.

figure number	pole angle	pole magnitude being plotted $\lvert a \rvert$
A-14	180°	0.0;2.0;4.0;6.0;8.0;10.0;12.0;14.0;16.0;18.0;20.0;22.0;
A-15	165°	0.0;2.0;4.0;6.0;8.0;10.0;12.0;14.0;16.0;
A-16	150°	0.0;2.0;4.0;6.0;8.0;10.0;12.0;
A-17	135°	0.0;2.0;4.0;6.0;8.0;10.0;
A-18	120°	0.0;2.0;4.0;6.0;8.0;
A-19	105°	0.0;2.0;4.0;6.0;
A-20	90°	0.0;1.0;2.0;3.0;4.0;
A-21	75°	0.0;1.0;2.0;3.0;
A-22	60°	0.0;1.0;2.0;
A-23	45°	0.0;1.0;2.0;
A-24	30°	0.00;0.25;0.50;0.75;1.00;1.25;1.50;
A-25	15°	0.00;0.25;0.50;0.75;1.00;
A-26	0°	0.00;0.25;0.50;0.75;

Table A-2: List of pole magnitudes plotted for the frequency response of T_2.

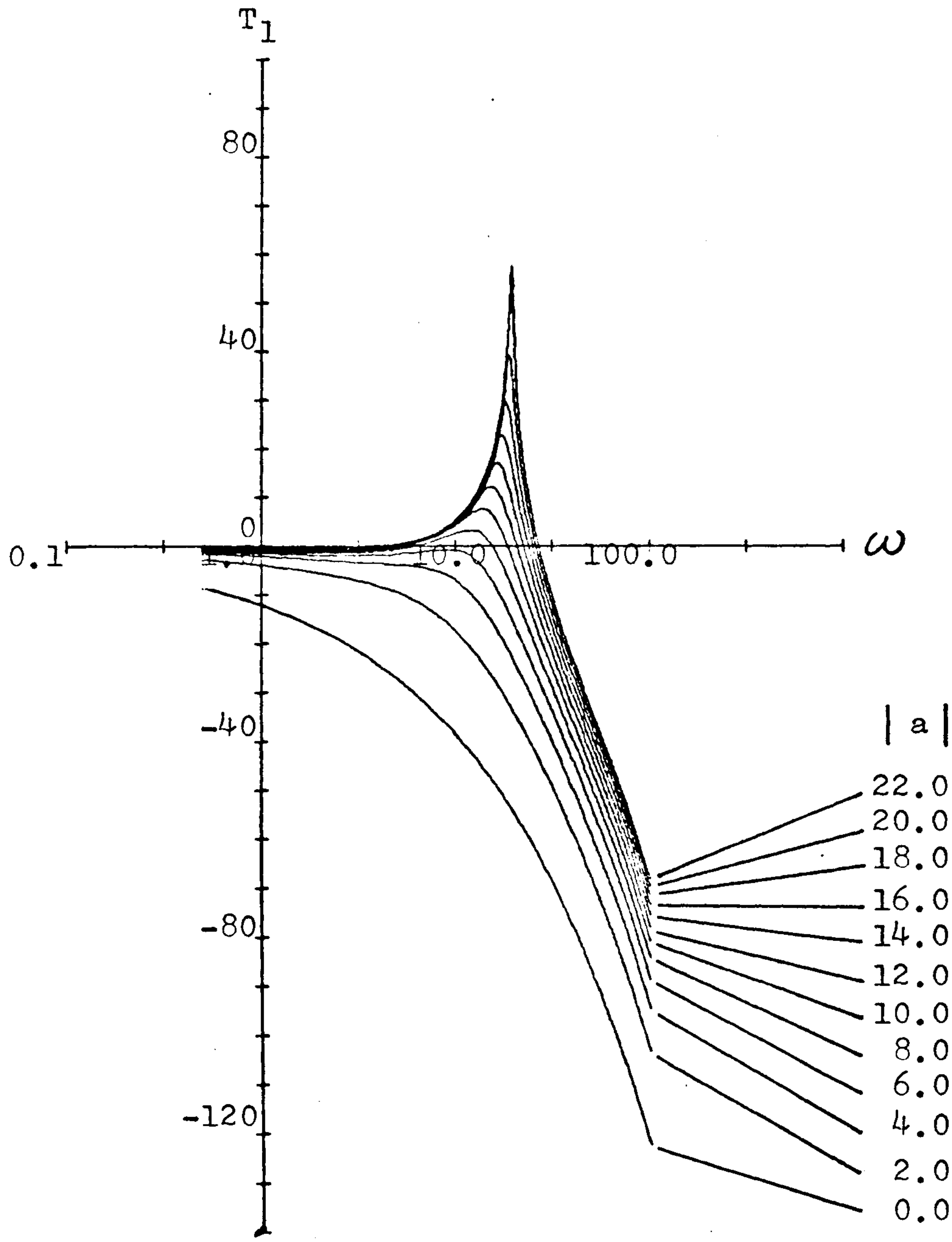


Figure A-1: Frequency response of T_1 with pole angle $\phi=180°$ in the w plane.

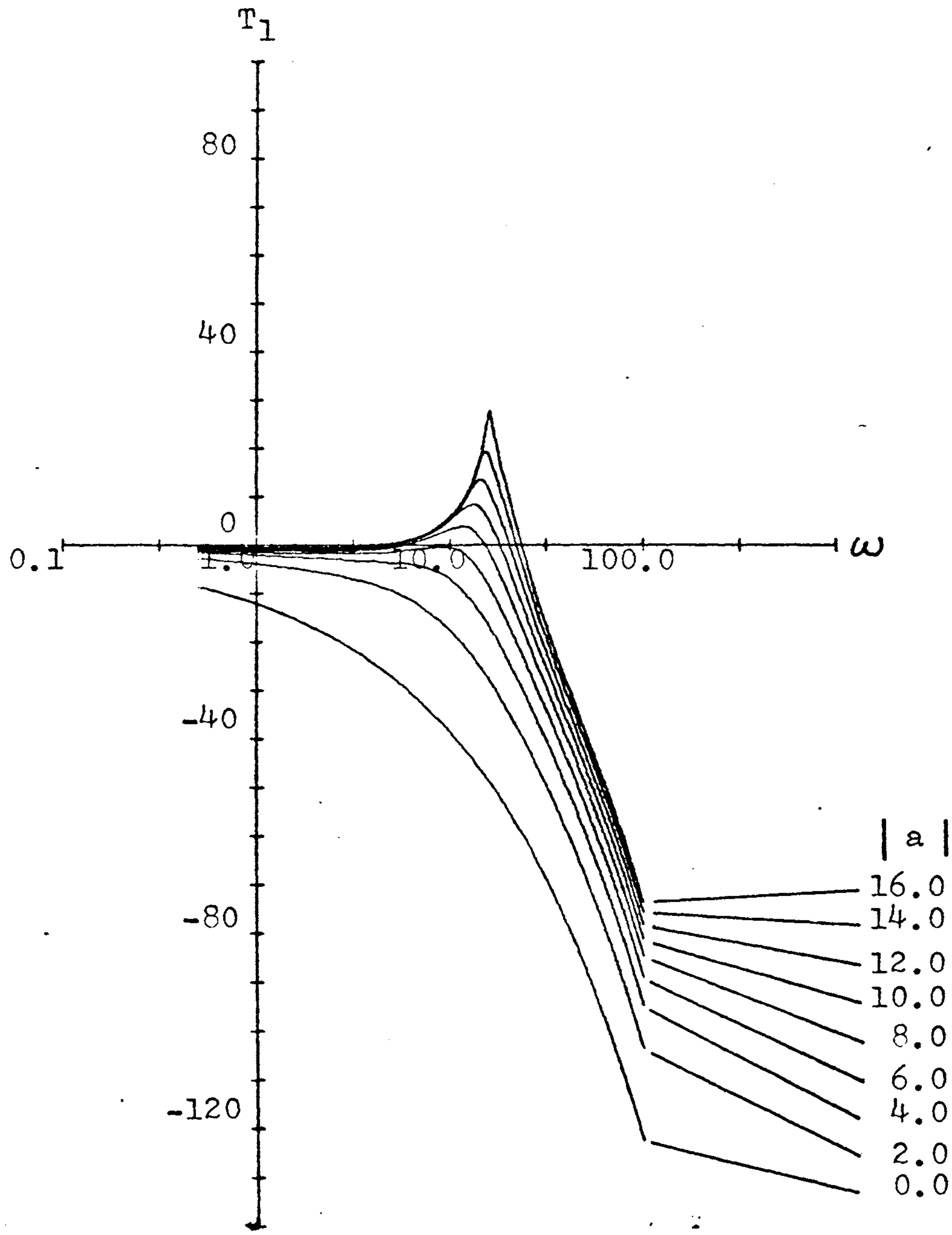


Figure A-2: Frequency response of T_1 with pole angle $\phi=165^\circ$ in the w plane.

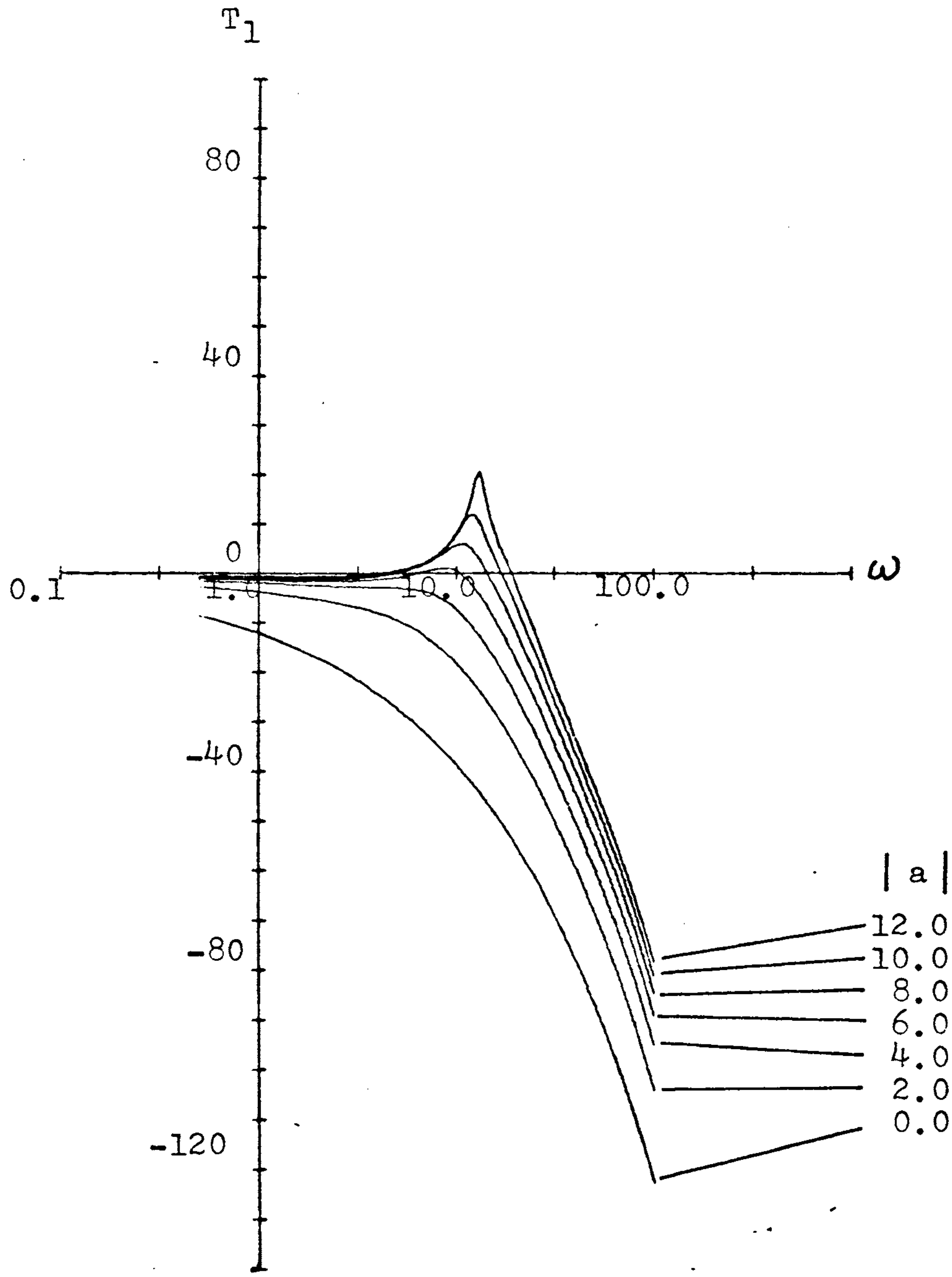


Figure A-3: Frequency response of T_1 with pole angle $\emptyset=150^{\circ}$ in the w plane.

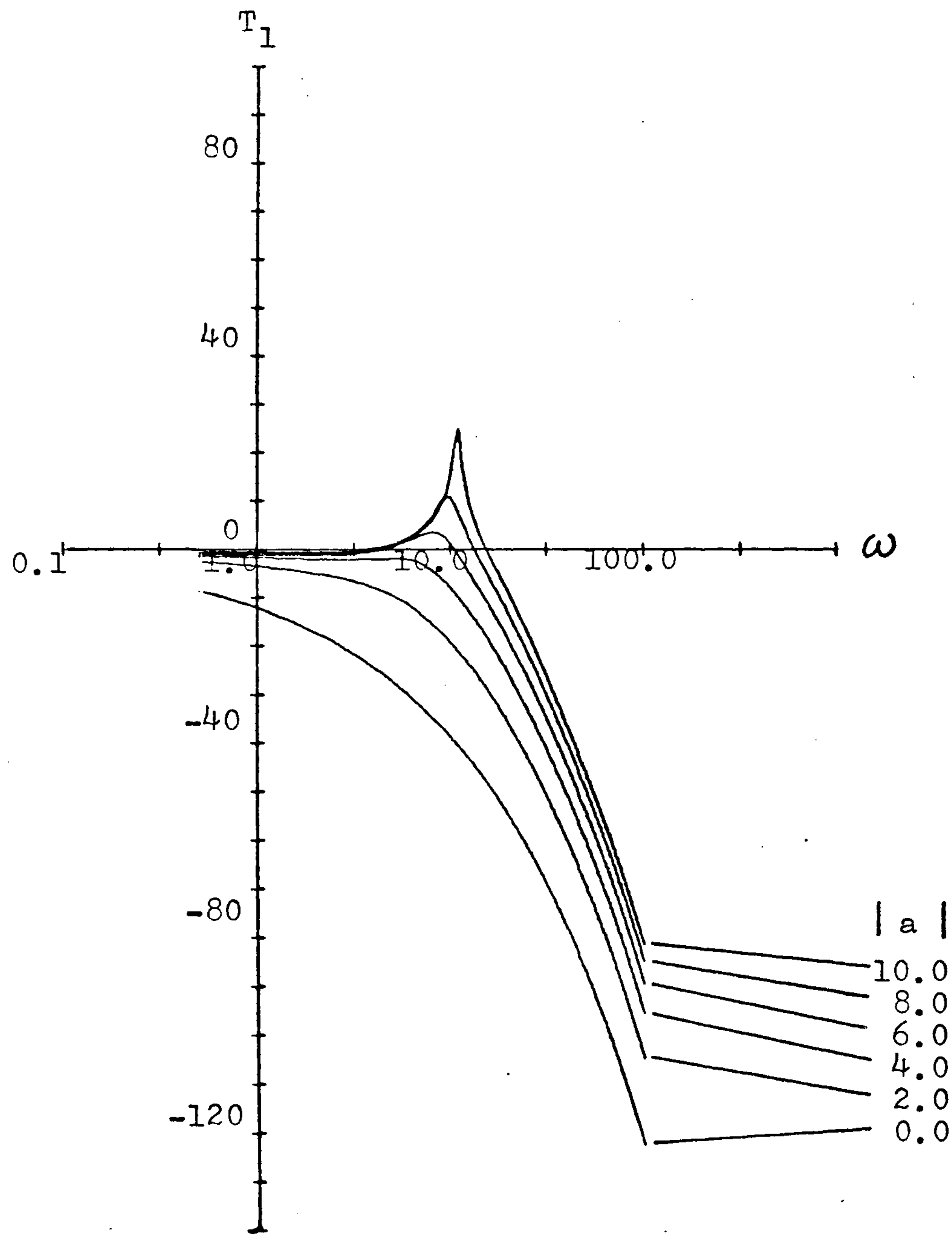


Figure A-4: Frequency response of T_1 with pole angle $\emptyset$=135° in the w plane.

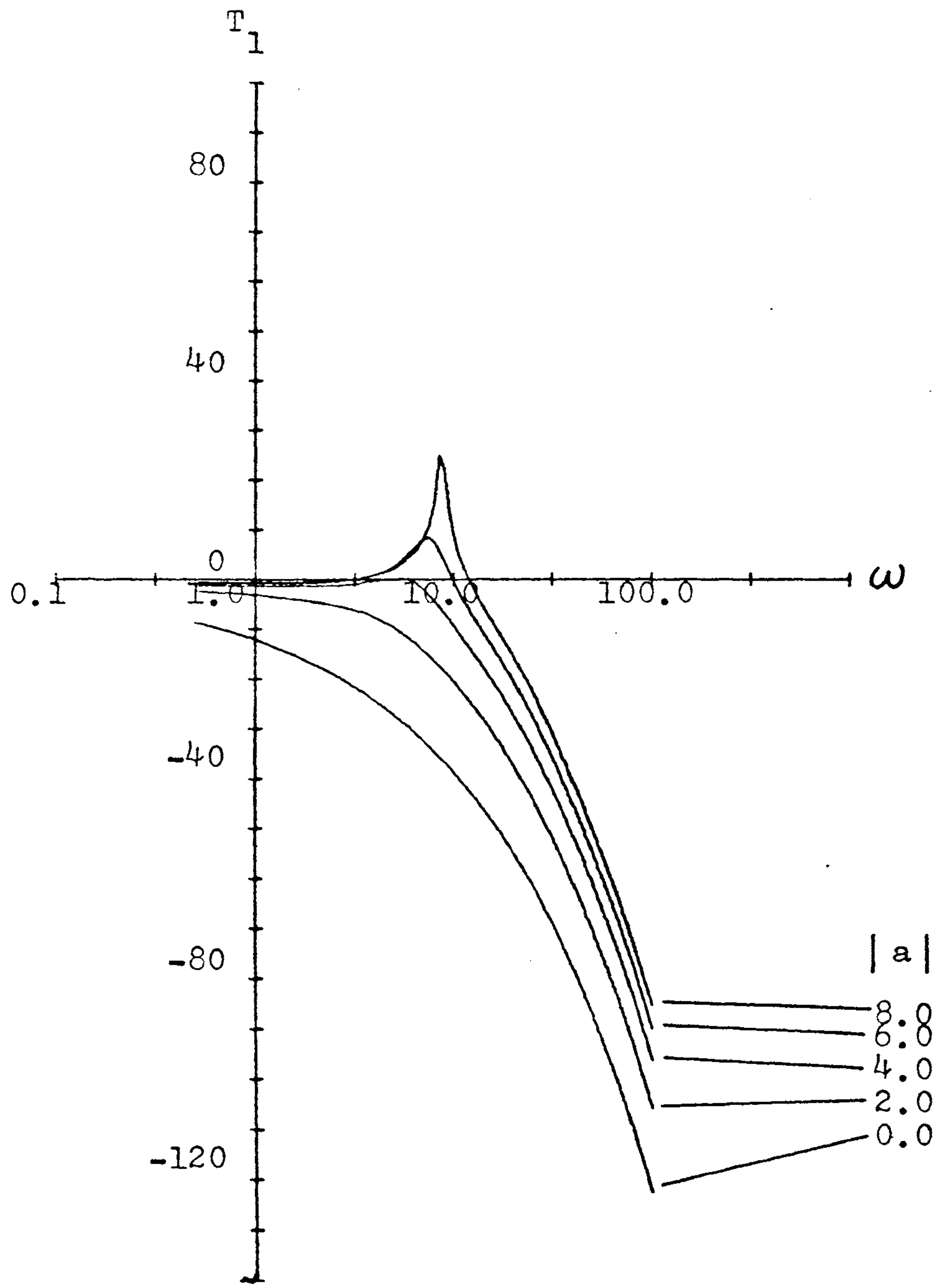


Figure A-5: Frequency response of T_1 with pole angle $\emptyset=120°$ in the w plane.

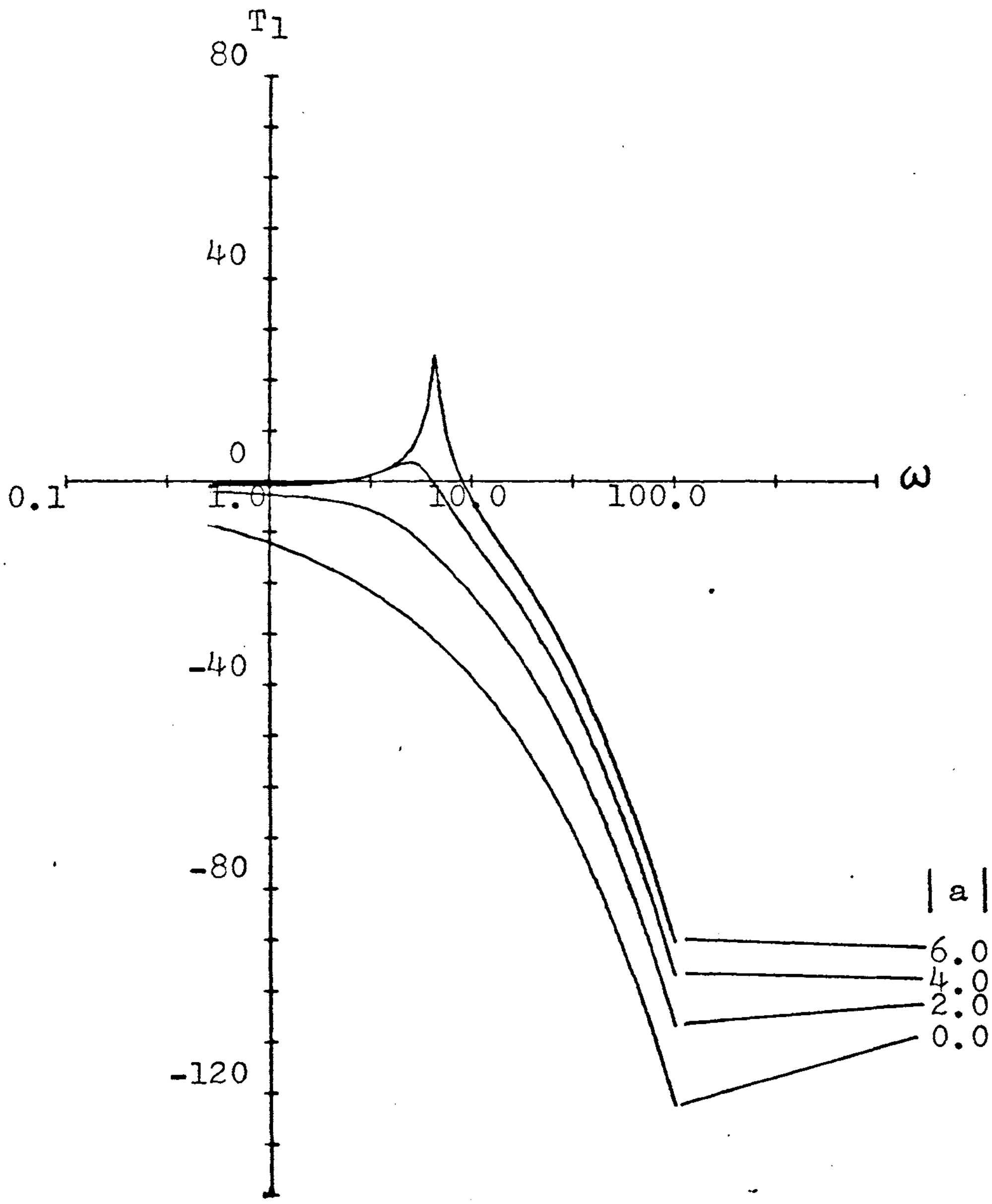


Figure A-6: Frequency response of T_1 with pole angle $\phi=105°$ in the w plane.

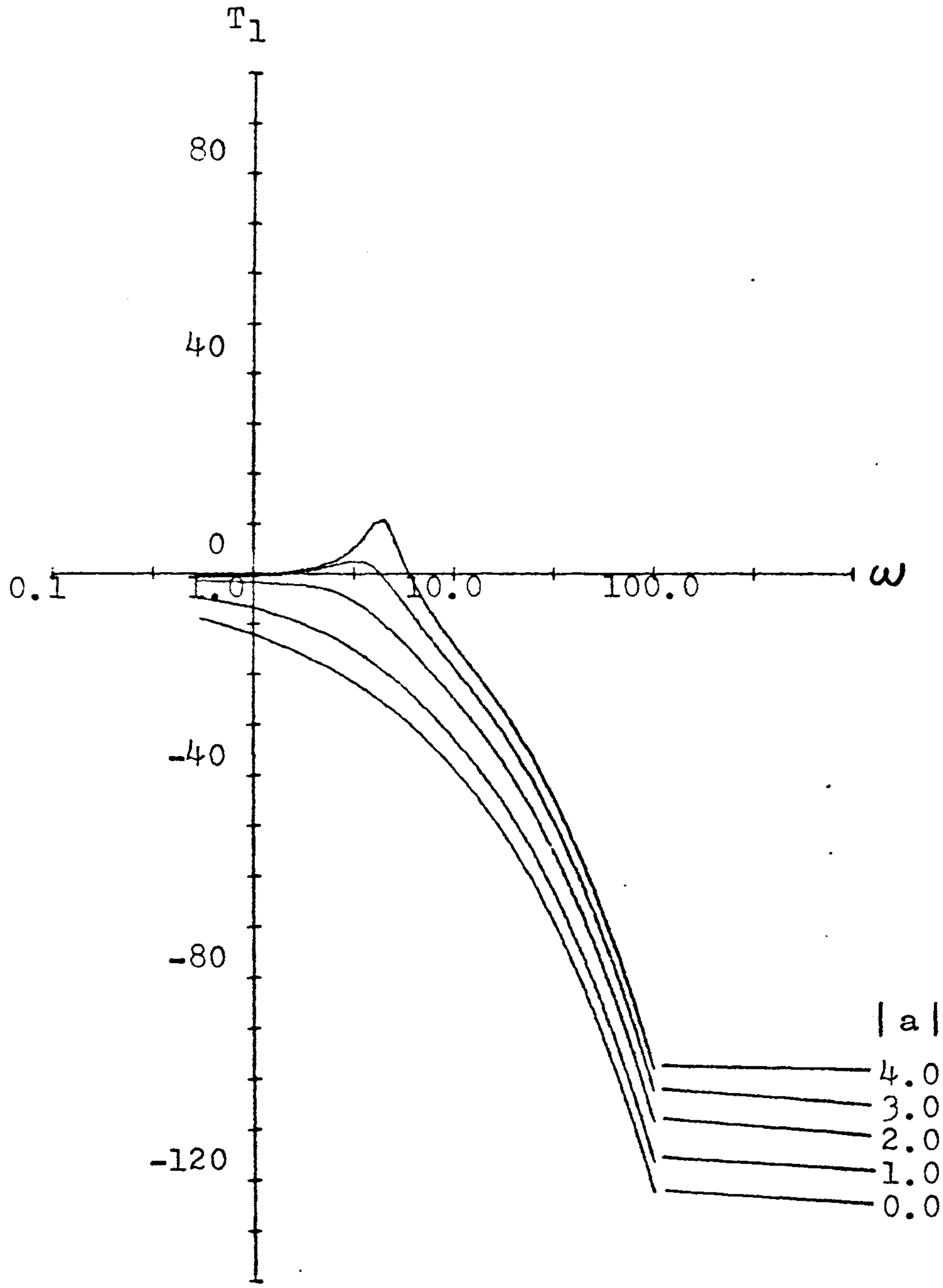


Figure A-7: Frequency response of T_1 with pole angle $\emptyset=90^\circ$ in the w plane.

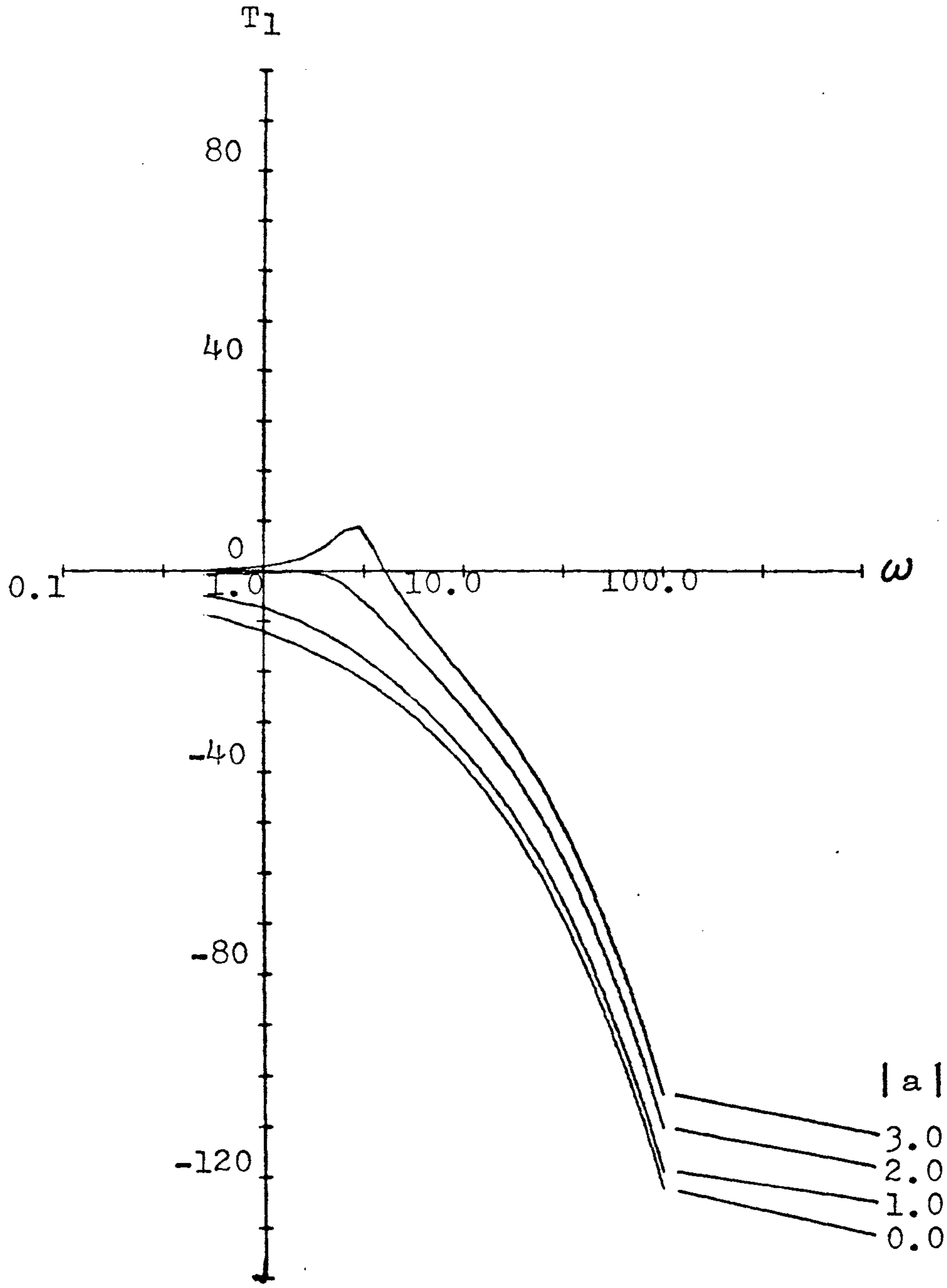


Figure A-8: Frequense response of T_1 with pole angle $\emptyset=75^\circ$ in the w plane.

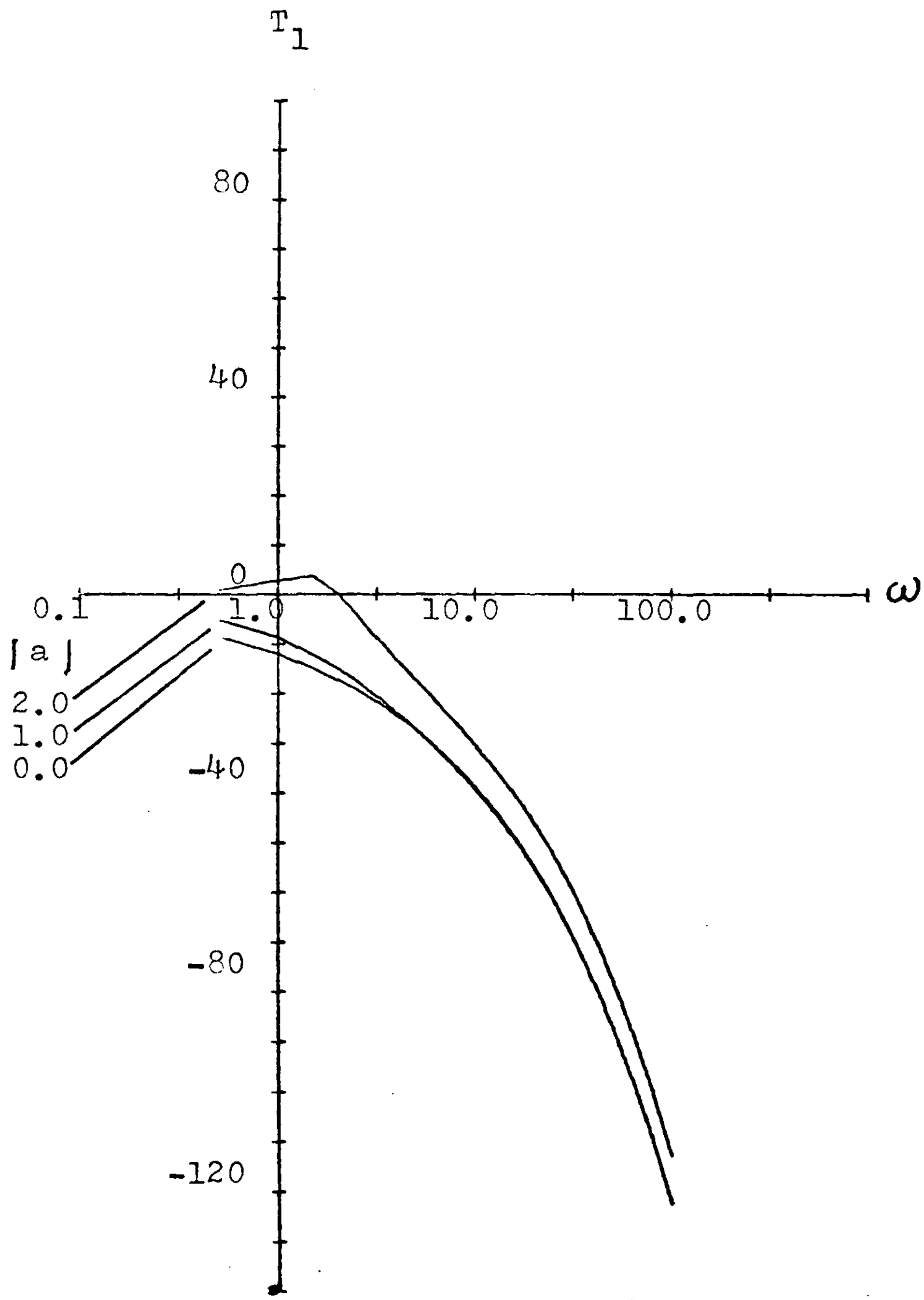


Figure A-9: Frequency response of T_1 with pole angle $\emptyset=60°$ in the w plane.

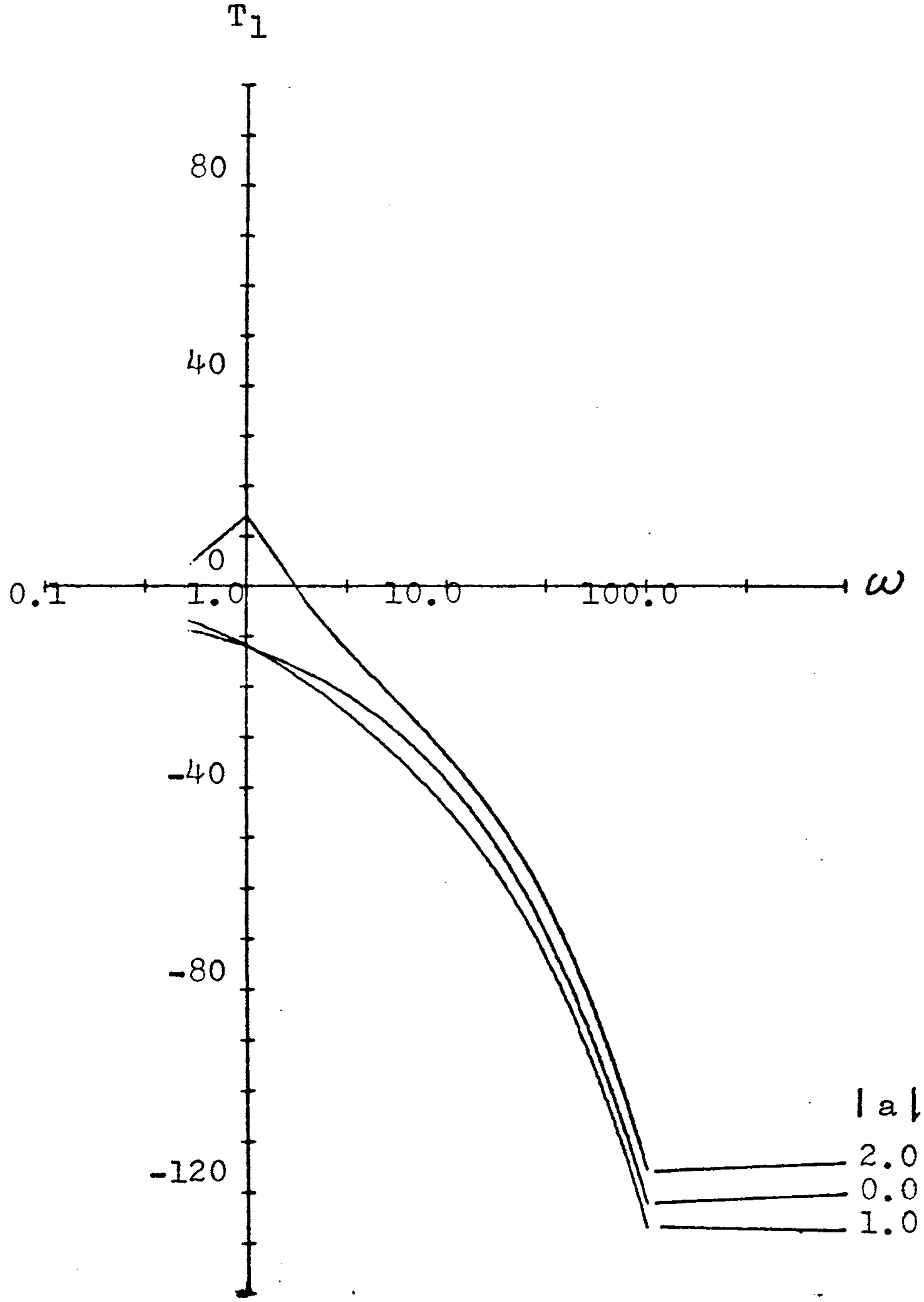


Figure A-10: Frequency response of T_1 with pole angle ϕ=45°
in the w plane.

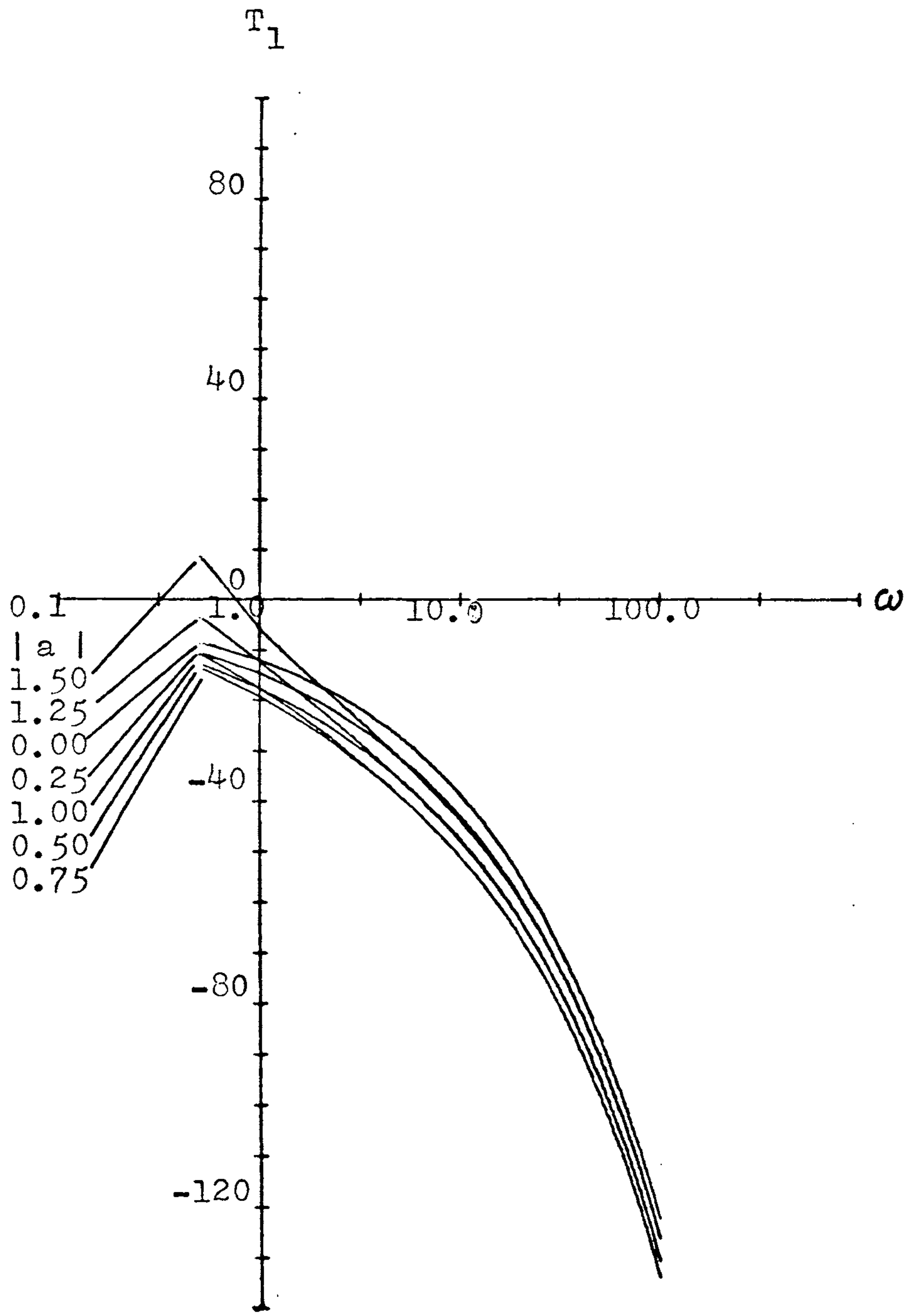


Figure A-11: Frequency response of T_1 with pole angle $\phi=30°$ in the w plane.

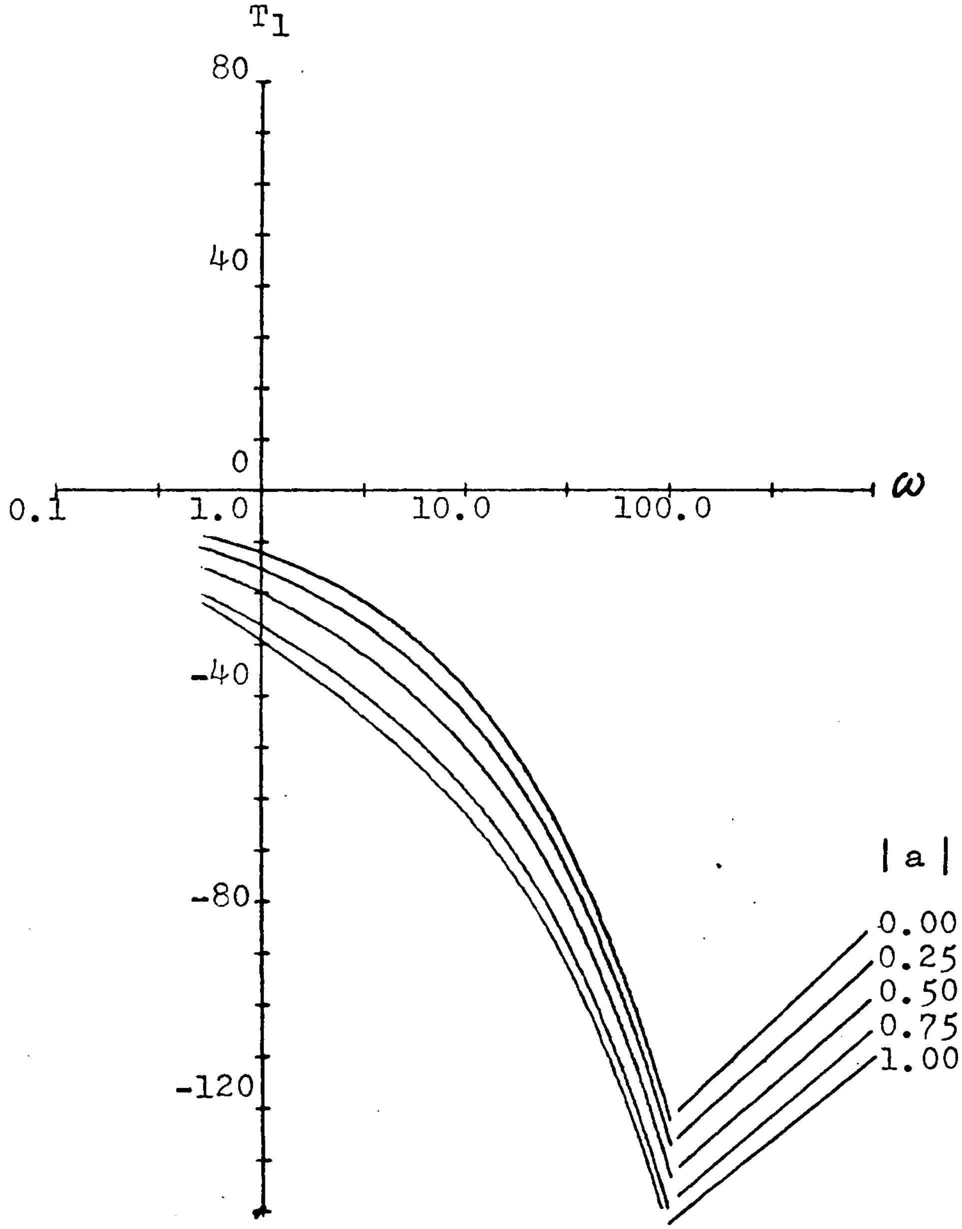


Figure A-12: Frequency response of T_1 with pole angle $\phi=15°$ in the w plane.

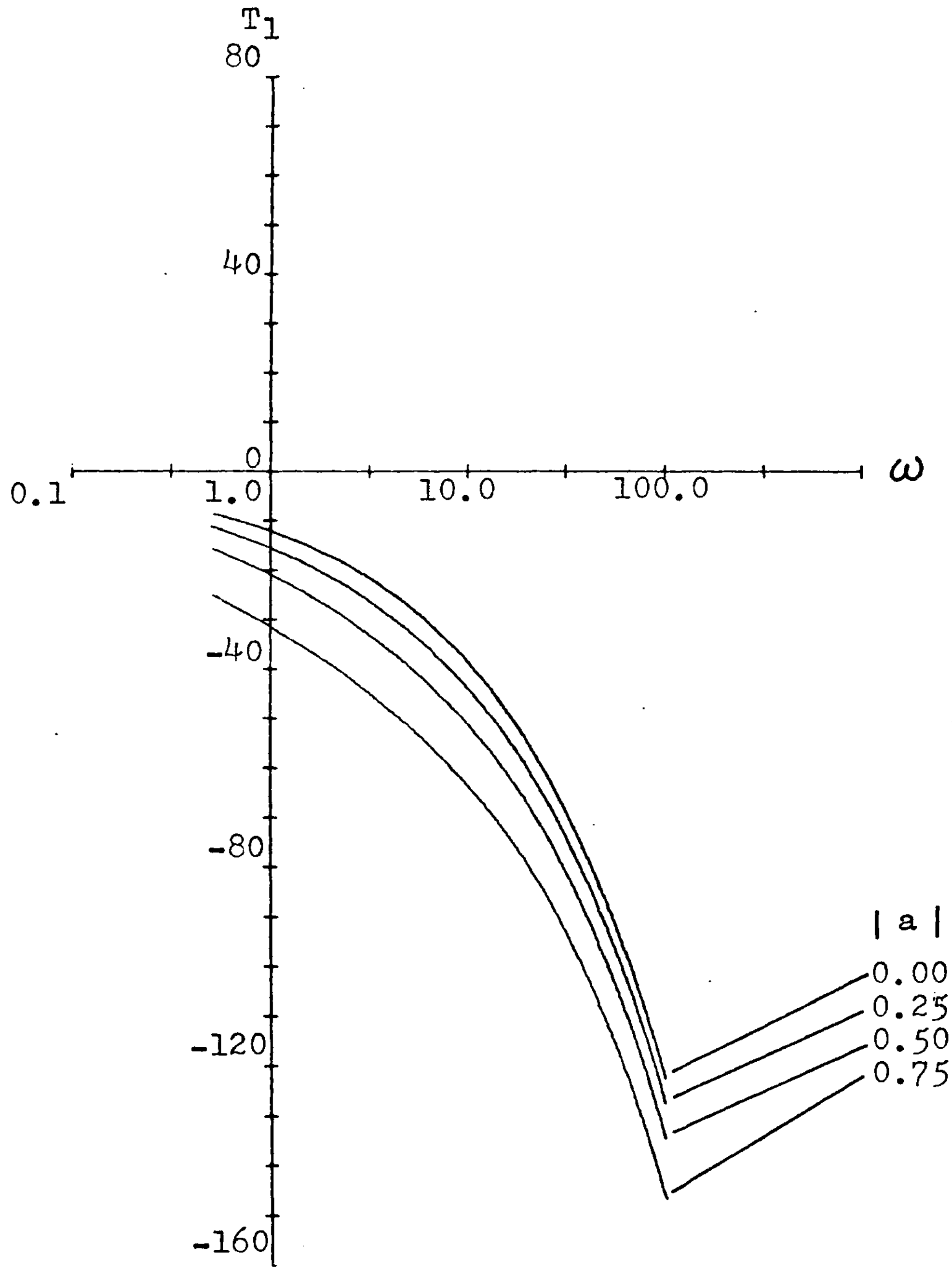


Figure A-13: Frequency response of T_1 with pole angle $\phi = 0°$ in the w plane.

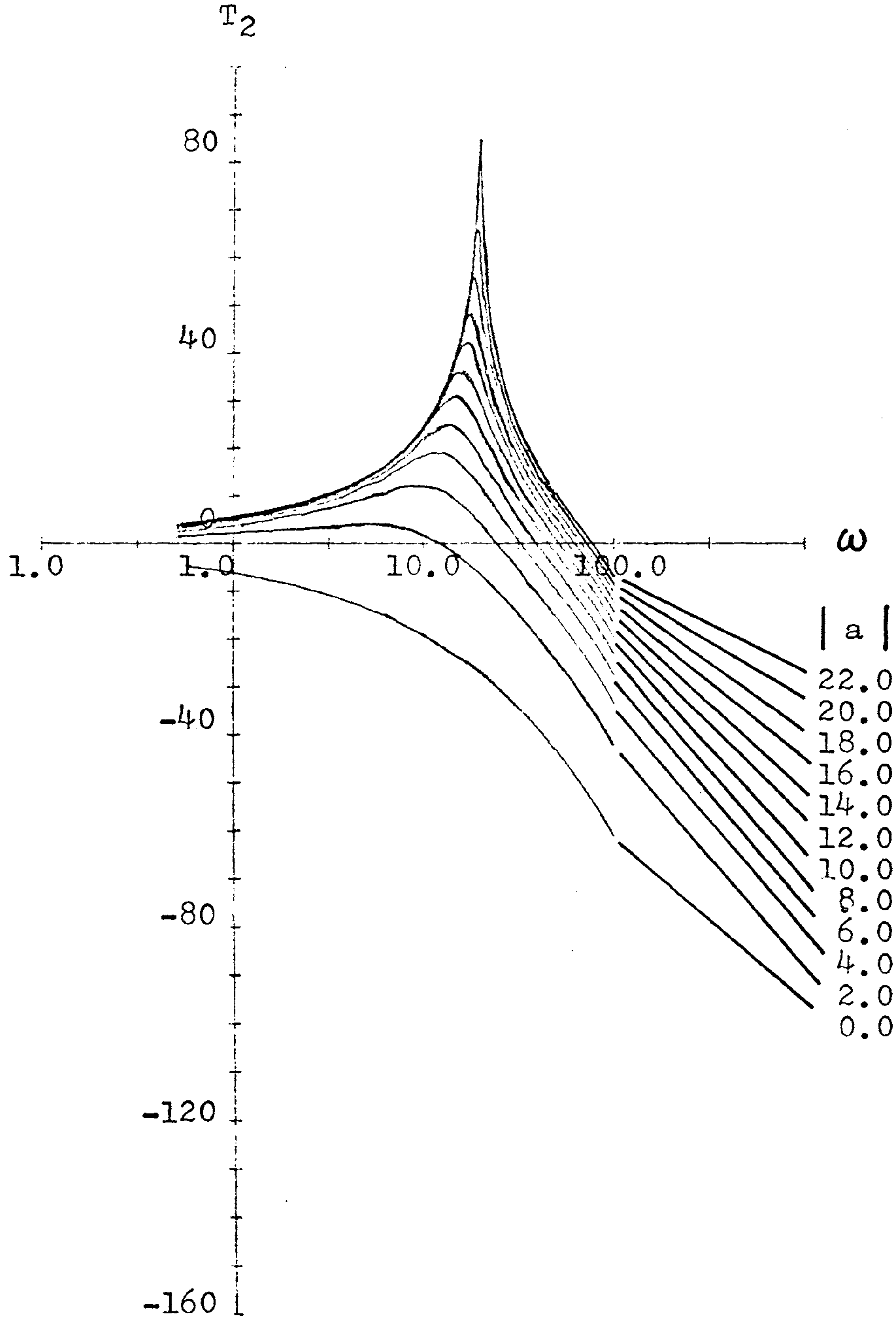


Figure A-14: Frequency response of T_2 with pole angle $\emptyset=180^\circ$ in the w plane.

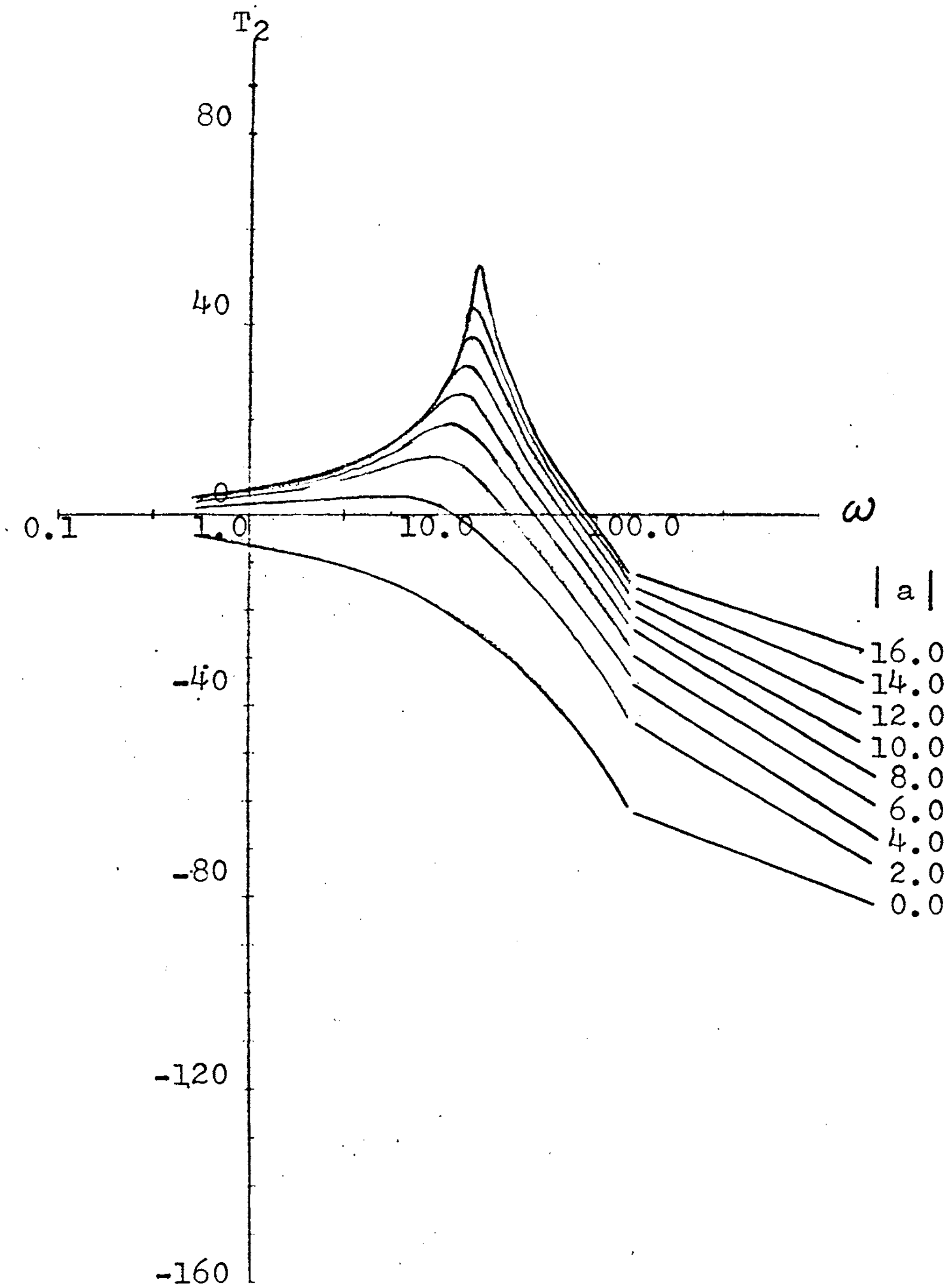


Figure A-15: Frequency response of T_2 with pole angle $\emptyset=165°$ in the w plane.

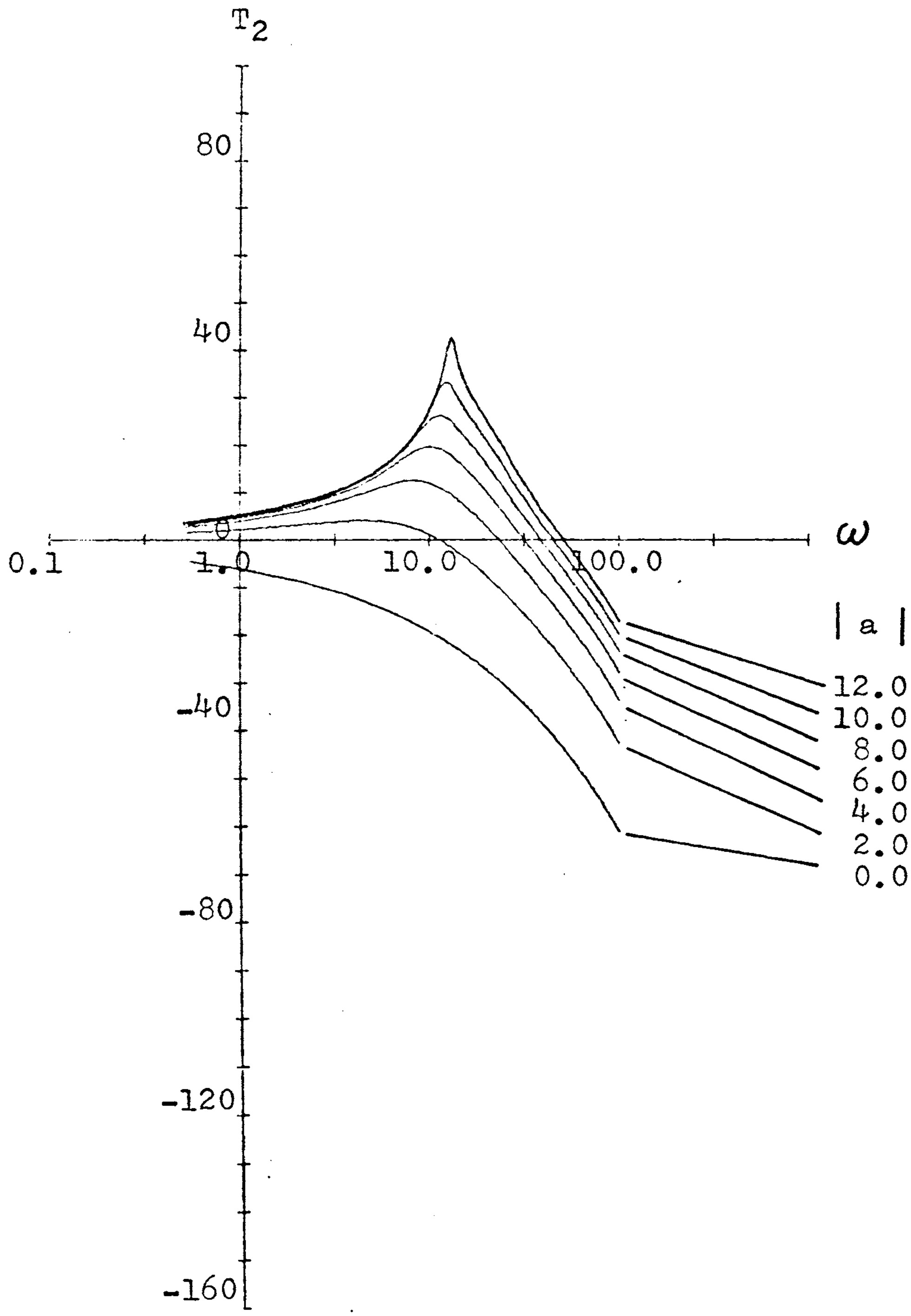


Figure A-16: Frequency response of T_2 with pole angle $\phi=150°$ in the w plane.

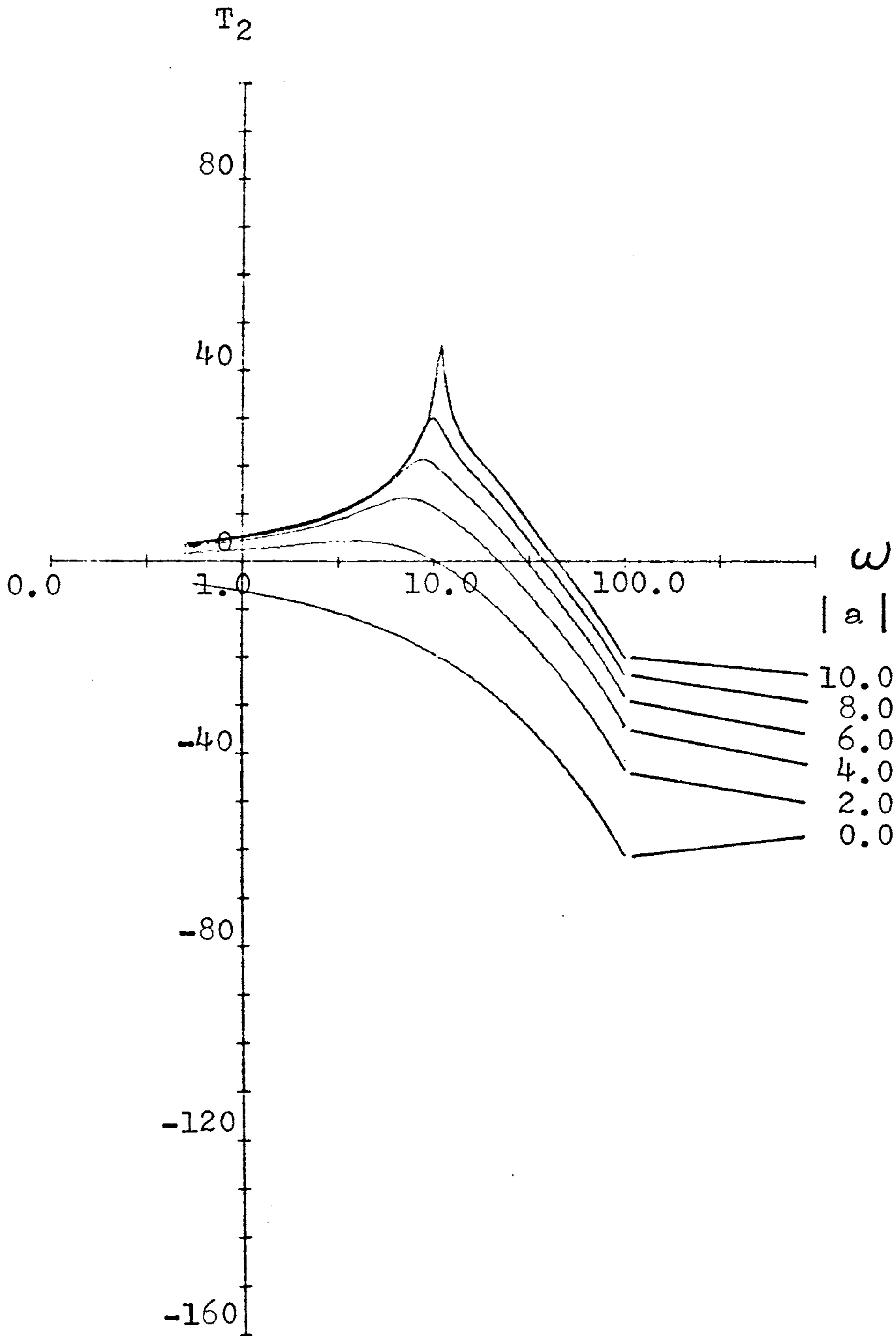


Figure A-17: Frequency response of T_2 with pole angle $\emptyset=135°$ in the w plane.

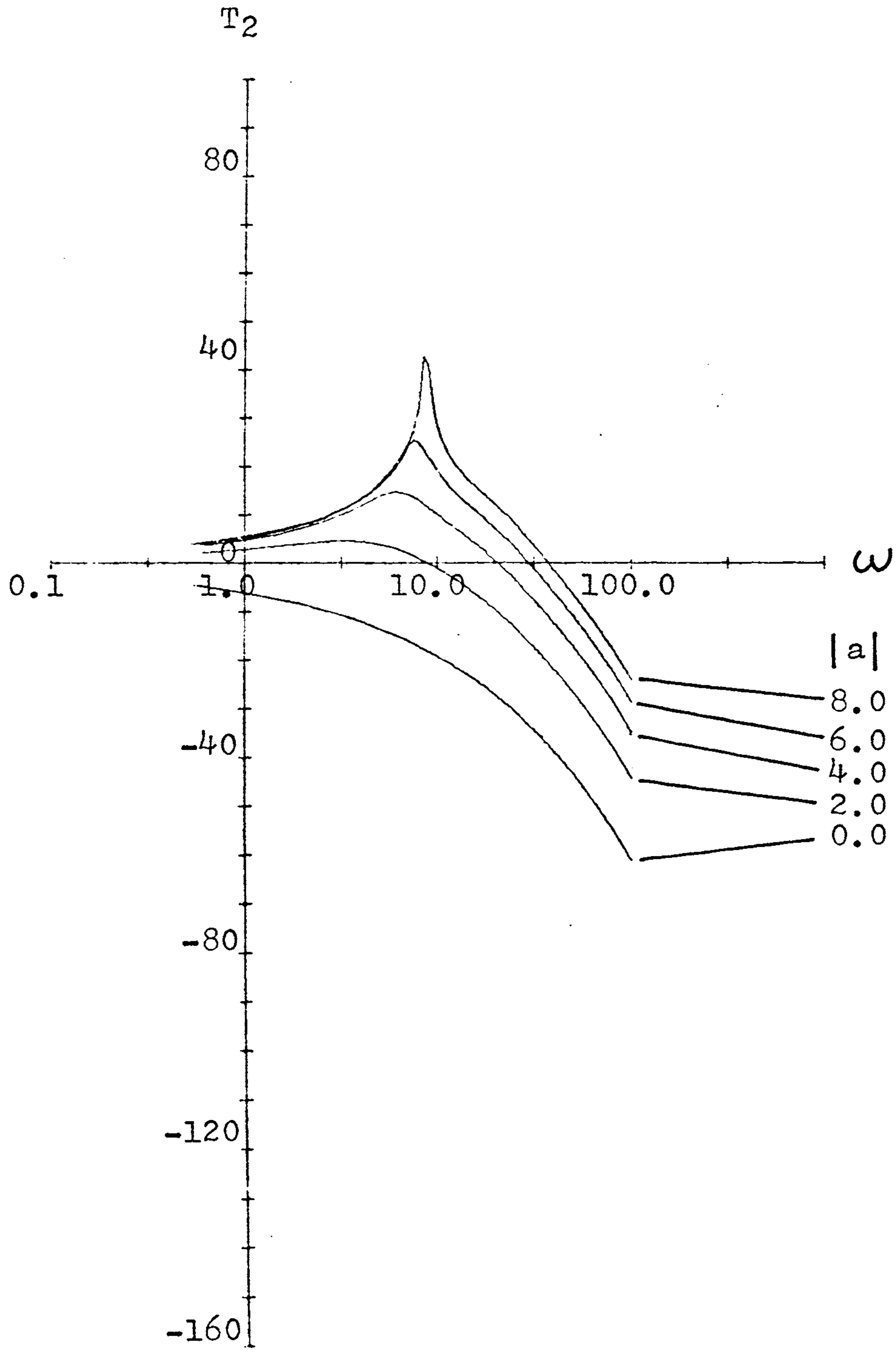


Figure A-18: Frequency response of T_2 with pole angle $\emptyset=120^\circ$ in the w plane.

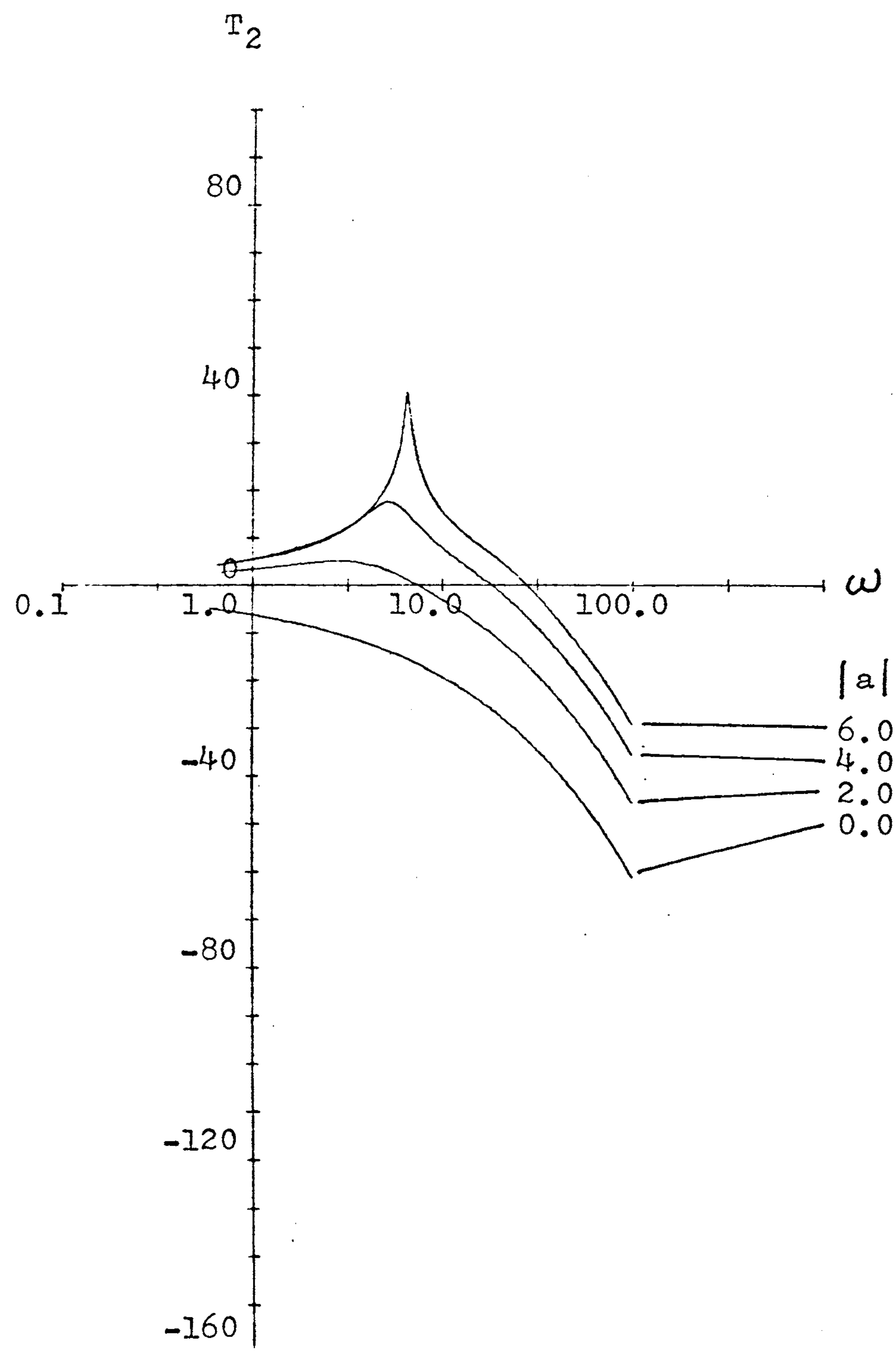


Figure A-19: Frequency response of T_2 with pole angle $\emptyset=105^\circ$ in the w plane.

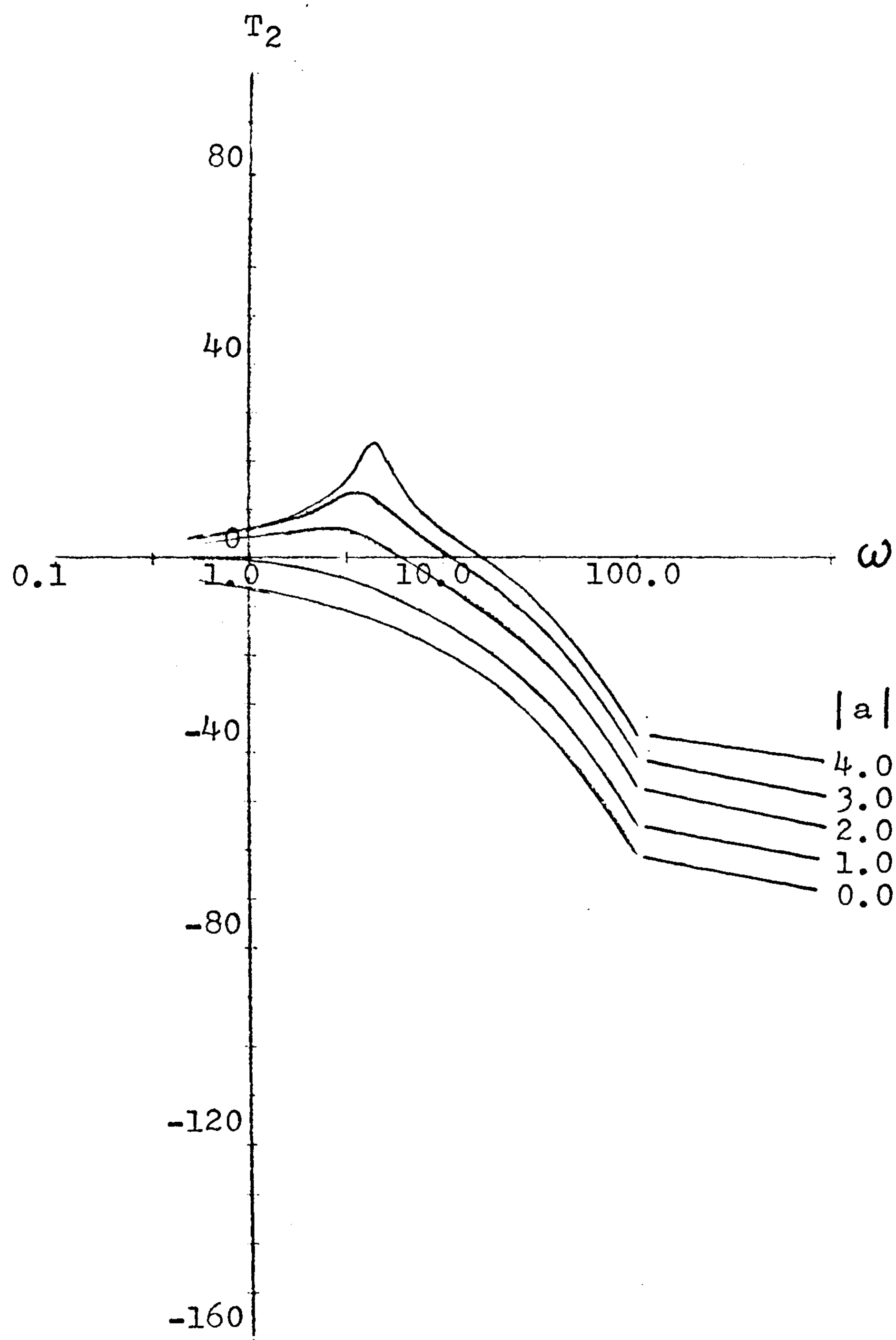


Figure A-20: Frequency response of T_2 with pole angle $\phi=90°$ in the w plane.

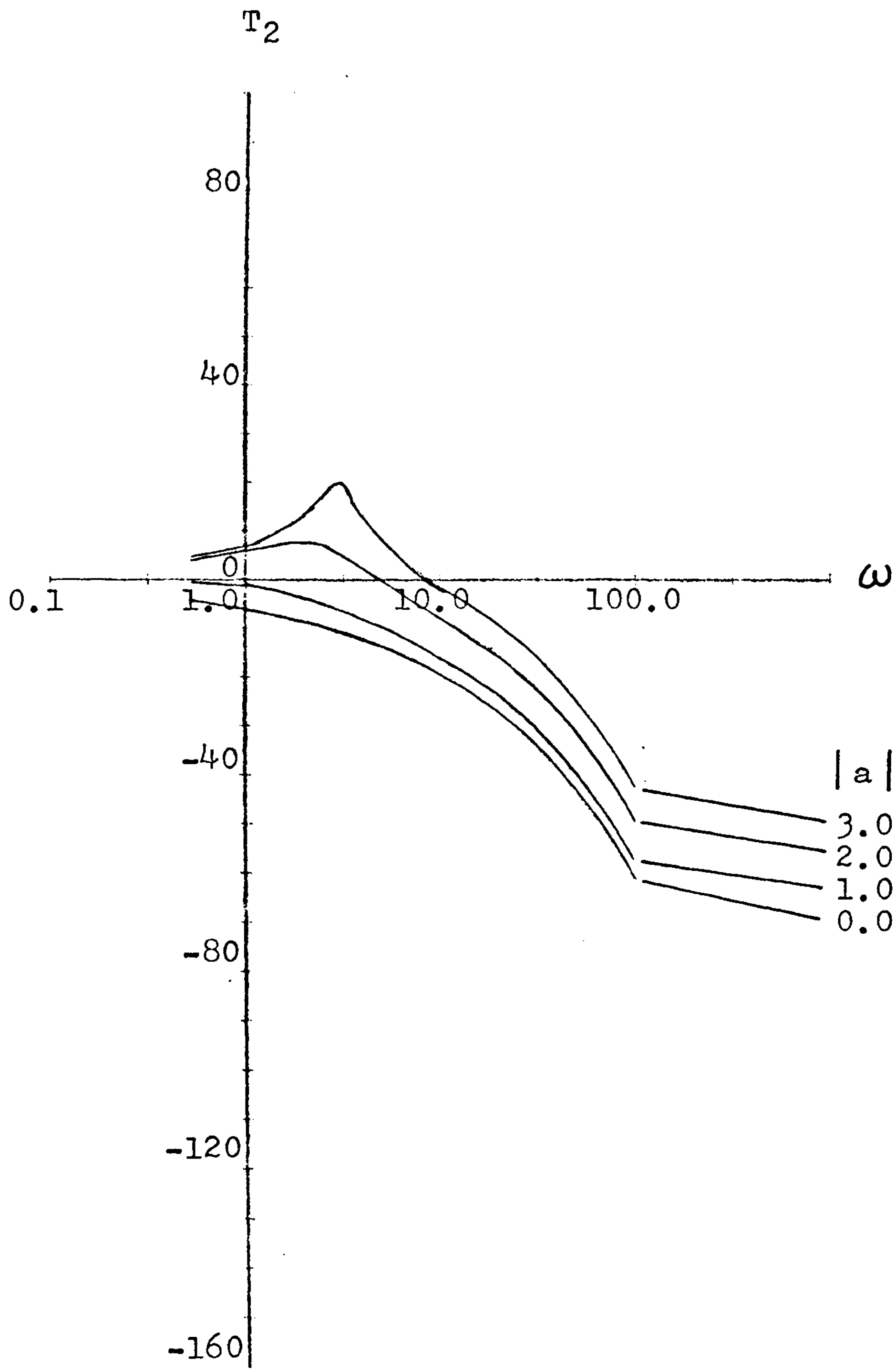


Figure A-21: Frequency response of T_2 with pole angle $\phi=75°$ in the w plane.

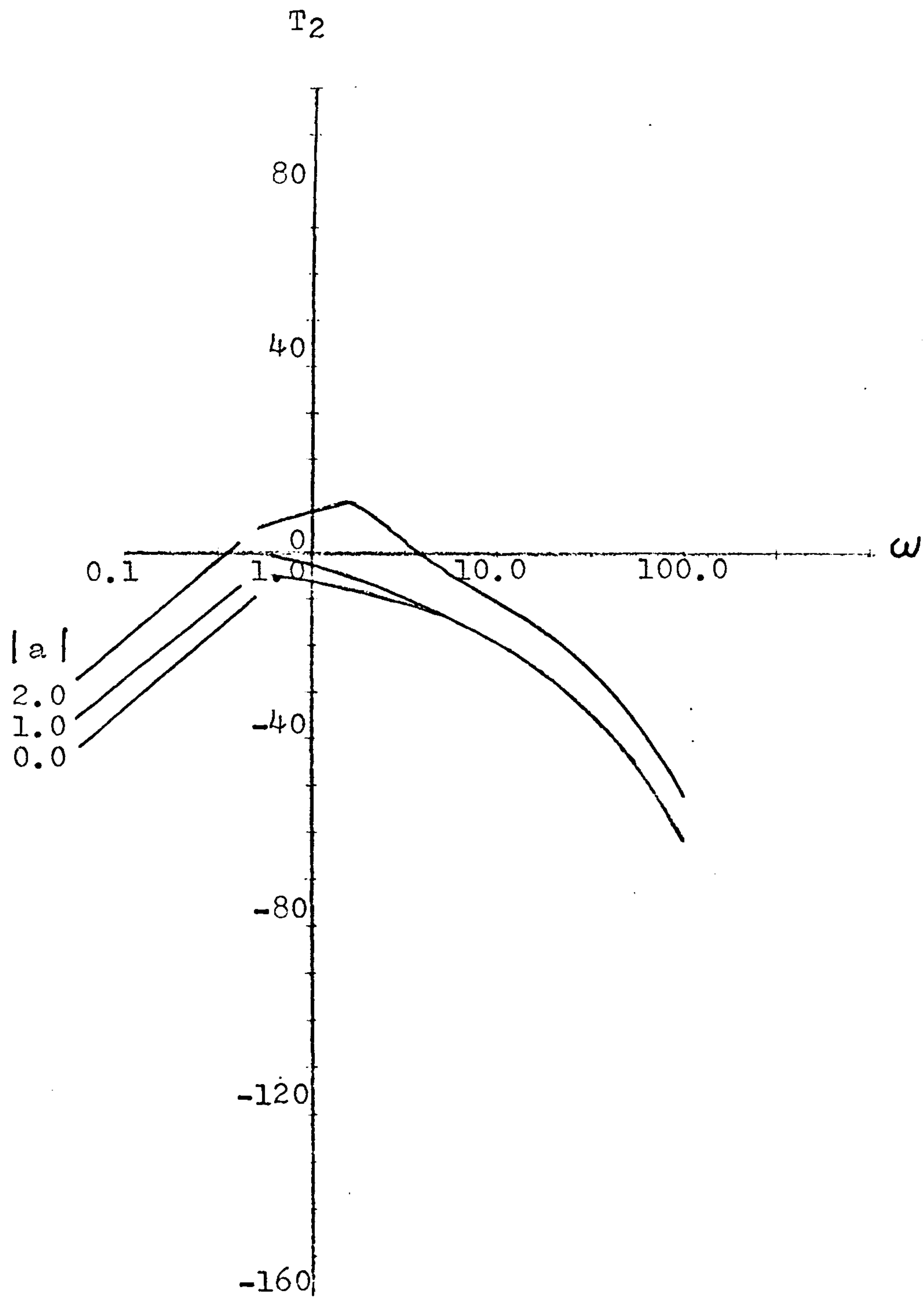


Figure A-22: Frequency response of T_2 with pole angle $\emptyset=60^\circ$ in the w plane.

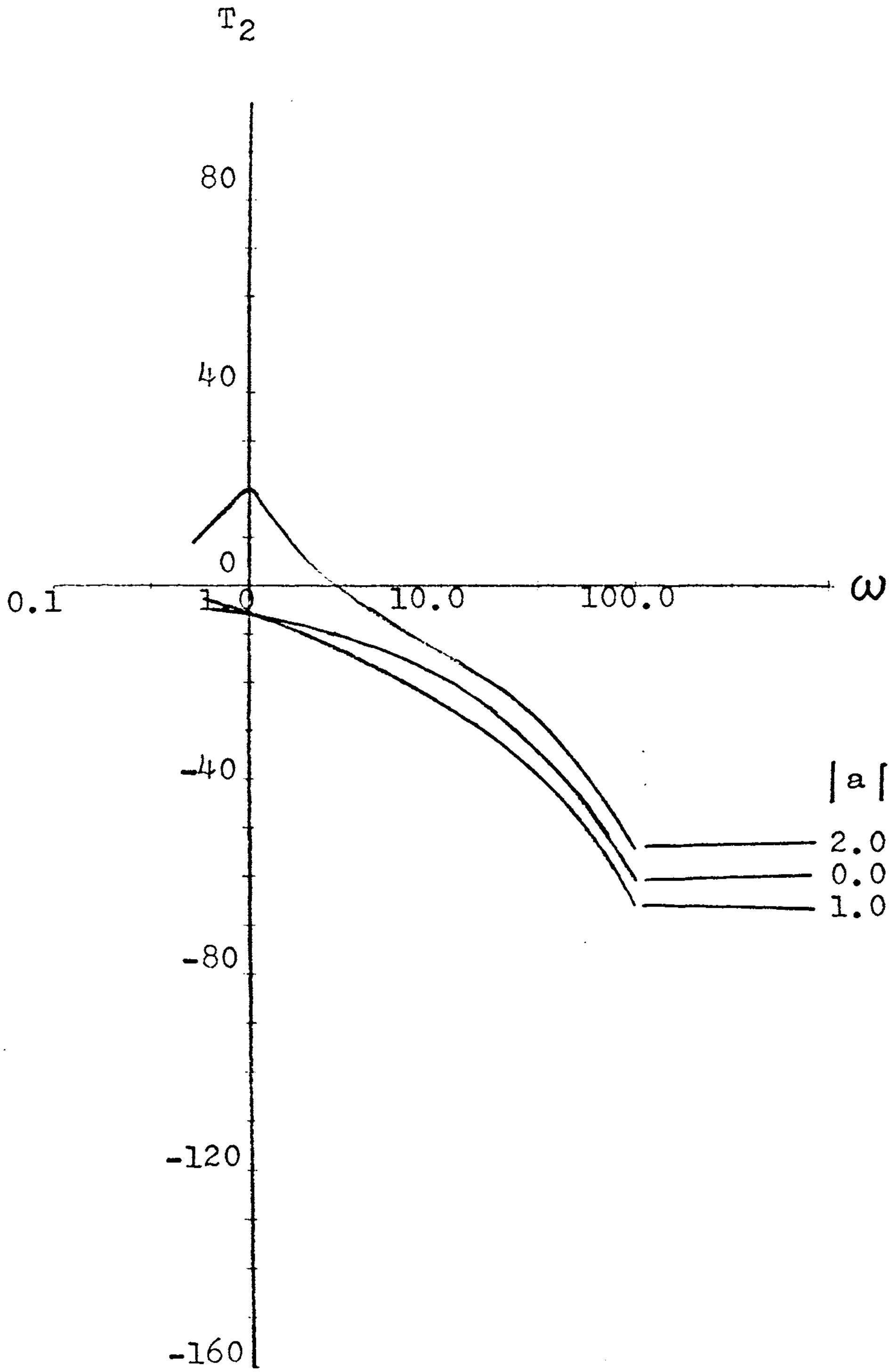


Figure A-23: Frequency response of T_2 with pole angle $\phi=45°$ in the w plane.

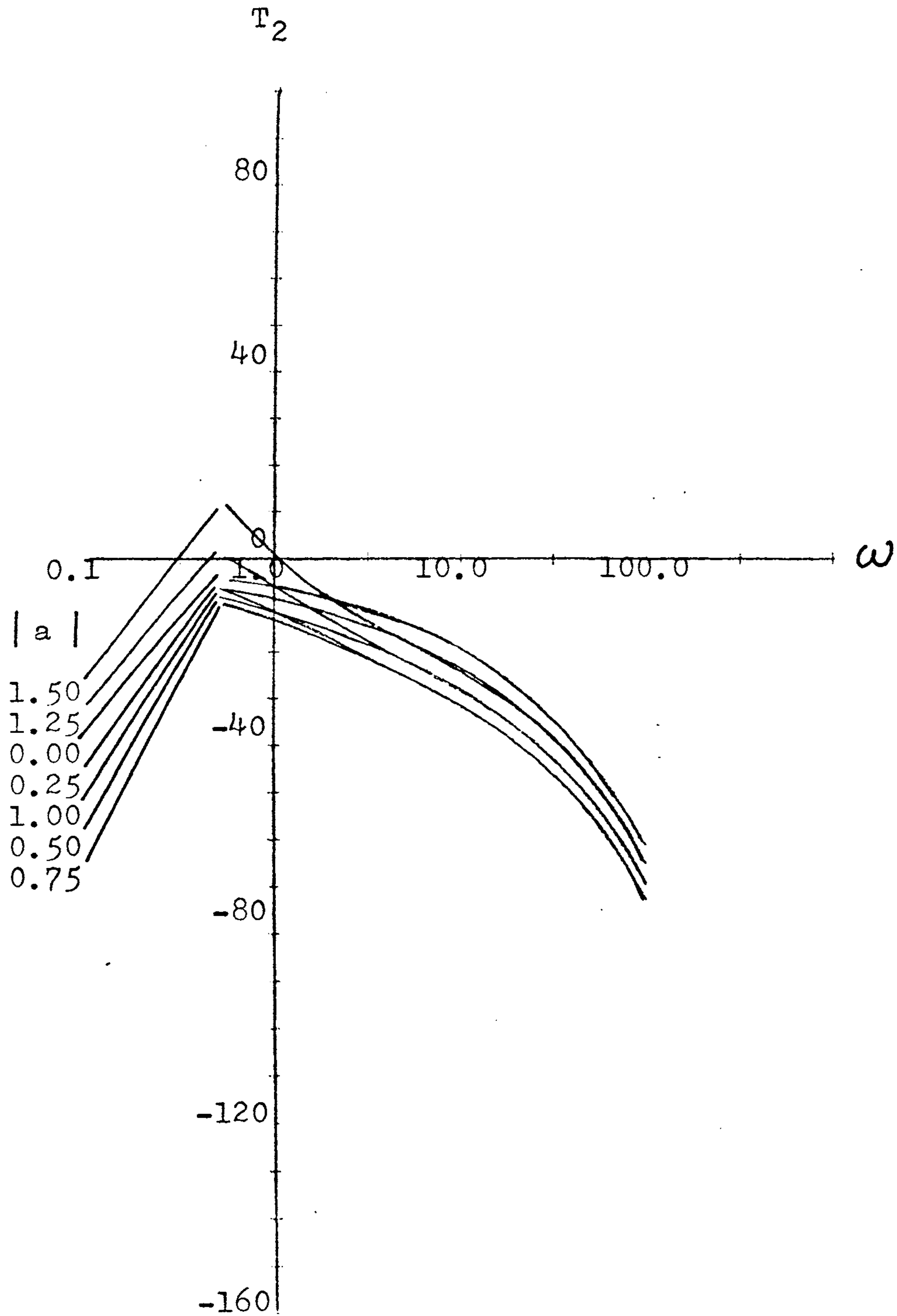


Figure A-24: Frequency response of T_2 with pole angle $\emptyset=30°$ in the w plane.

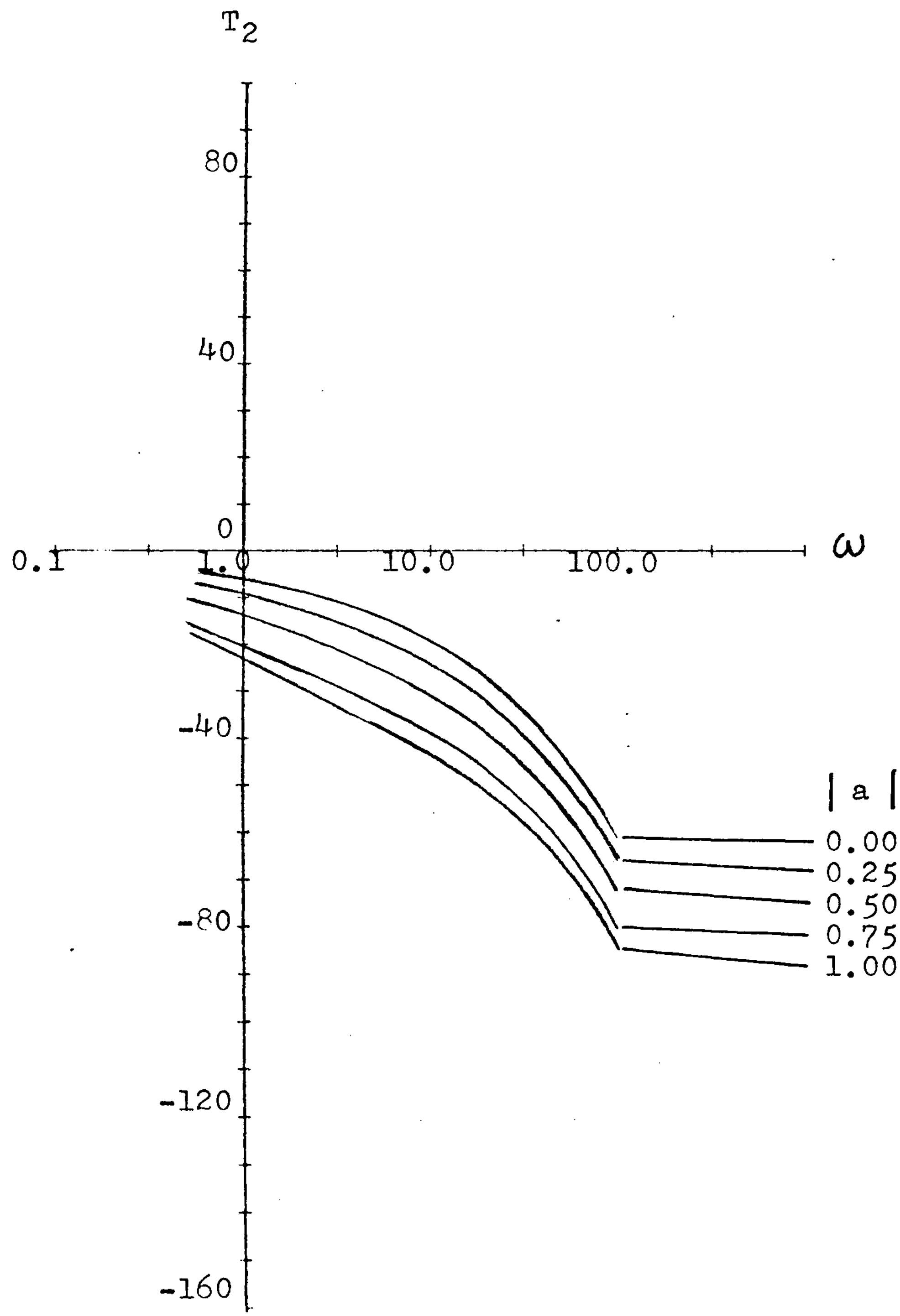


Figure A-25: Frequency response of T_2 with pole angle $\emptyset=15°$ in the w plane.

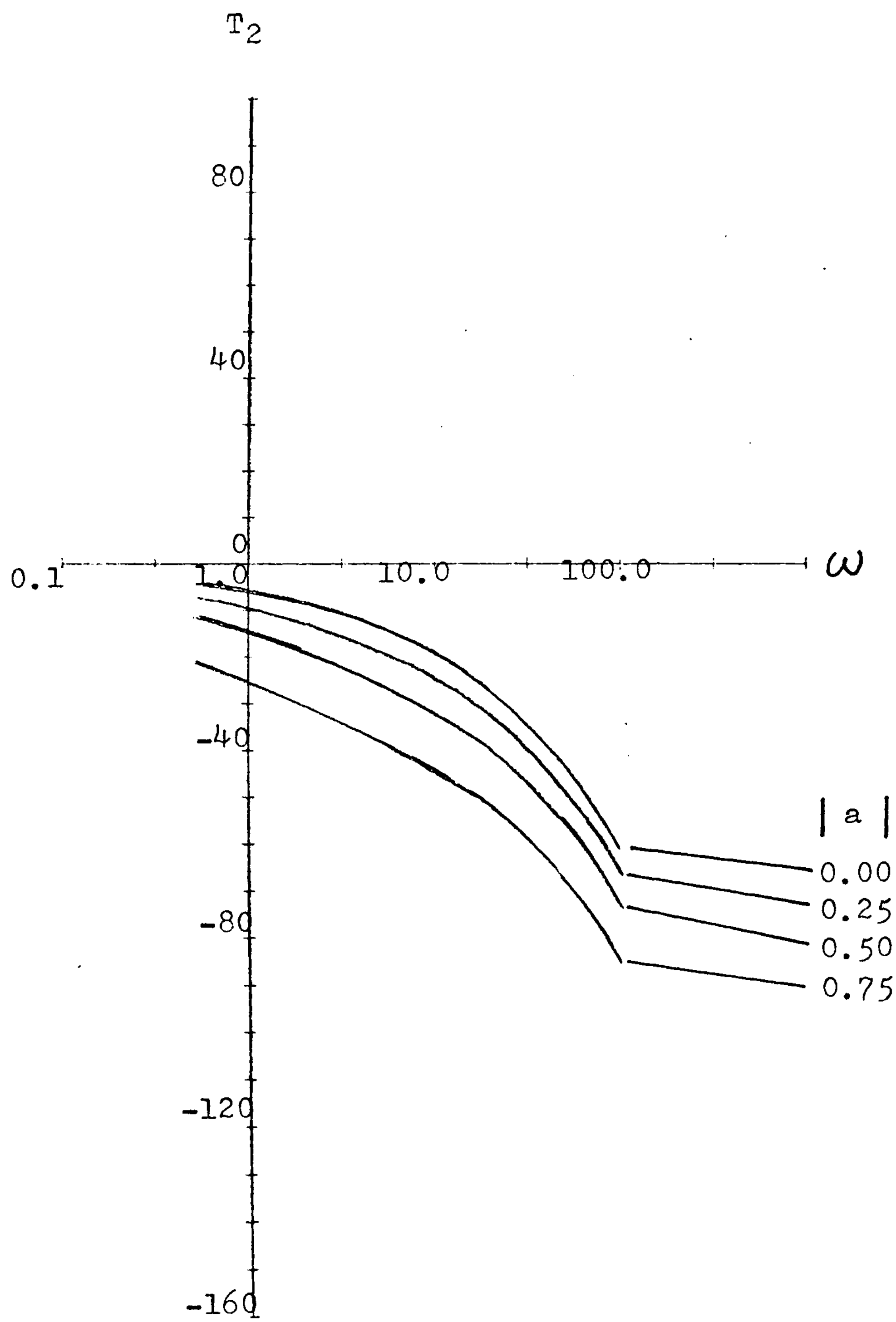


Figure A-26: Frequency response of T_2 with pole angle $\emptyset=0°$ in the w plane.

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